

HIGHER TOPOLOGICAL COMPLEXITY AND HOMOTOPY DIMENSION OF CONFIGURATION SPACES ON SPHERES

IBAI BASABE, JESÚS GONZÁLEZ, YULI B. RUDYAK, AND DAI TAMAKI

ABSTRACT. Yu. Rudyak has recently extended Farber's notion of topological complexity by defining, for $n \geq 2$, the n^{th} topological complexity $\text{TC}_n(X)$ of a path-connected space X —Farber's original notion is recovered for $n = 2$. In this paper we develop further the properties of this extended concept, relating it to the Lusternik-Schnirelmann category of cartesian powers of X , as well as to the cup-length of the diagonal embedding $X \hookrightarrow X^n$. We compute the numerical values of TC_n for products of spheres, closed 1-connected symplectic manifolds (e.g. complex projective spaces), and quaternionic projective spaces. We explore the symmetrized version of the concept ($\text{TC}_n^S(X)$) and introduce a new symmetrization ($\text{TC}_n^\Sigma(X)$) which is a homotopy invariant of X .

We obtain a (conjecturally sharp) upper bound for $\text{TC}_n^S(X)$ when X is a sphere. This is attained by introducing and studying the idea of *cellular stratified spaces*, a new concept that allows us to import techniques from the theory of hyperplane arrangements in order to construct finite CW complexes of the lowest possible dimension modelling, up to equivariant homotopy, configuration spaces of ordered distinct points on spheres—our models are in fact simplicial complexes. In particular, we show that the configuration space of n points (either ordered or unordered) in the k -dimensional sphere has homotopy dimension $(n-1)(k-1)+1$.

2010 Mathematics Subject Classification: Primary 52C35, 55M30, 55R80, 57Q05. Secondary 14N20, 55R05, 57Q40, 68T40.

Key words and phrases: Lusternik-Schnirelmann category; Schwarz genus; topological complexity; configuration spaces; hyperplane arrangements; Salvetti complex; cellular stratified spaces; homotopy dimension.

CONTENTS

1. Introduction	2
2. Preliminaries	7
3. Properties of Higher Topological Complexity	9
4. Symmetric Topological Complexity	14
5. Bounding genus (ε_n) for Spheres	19
6. Cellular Stratifications	21
6.1. Basic properties	22
6.2. Barycentric subdivision	28
7. The Σ_n -Equivariant Homotopy Model of $C_n(S^k)$	37
7.1. Braid stratifications of euclidean spaces	37
7.2. Braid stratifications of products of a sphere	40

1. INTRODUCTION

The concept of topological complexity “TC” was introduced by Michael Farber motivated by the most basic problem of robot motion planning: finding the smallest number of continuous instructions for a robot to move from one point to another in a path-connected space.

In greater detail, given a mechanical system S , a *motion planning algorithm* for S is a rule that assigns a (continuous) motion from A to B to each pair (A, B) of positions of S , [La91, LV06]. Let X denote the configuration space of S ; thus the positions of S are the points of X , and a motion from position A to position B of S is a (continuous) path in X starting on A and finishing on B .

Let PX denote the set of all paths $\gamma : [0, 1] \rightarrow X$, topologized with the compact-open topology. We will denote by $\pi : PX \rightarrow X \times X$ the map associating to any path $\gamma \in PX$ the pair of its initial and final points, $\pi(\gamma) = (\gamma(0), \gamma(1))$. In these terms, a motion planning algorithm is a map $s : X \times X \rightarrow PX$ such that $\pi \circ s = \text{id}_{X \times X}$, i.e. a section of π .

It is easy to see that a *continuous* motion planning algorithm (i.e., a continuous section s of π) exists only for X contractible. So, it is pertinent to express $X \times X$ as a union of subsets, each of which admits a continuous motion planning algorithm (such a collection of subsets of $X \times X$ and corresponding sections is called a *motion planner* on X). To formalize this idea, Michael Farber [Fa03] defined the topological complexity of a space as follows:

Definition. Given a path-connected topological space X , the *topological complexity* of X for the robot motion planning problem, $\text{TC}(X)$, is the least number k such that the cartesian product $X \times X$ can be covered by k open subsets

$$X \times X = U_1 \cup U_2 \cup \cdots \cup U_k$$

such that for any $i = 1, 2, \dots, k$ there exists a continuous motion planning algorithm $s_i : U_i \rightarrow PX$, $\pi \circ s_i = \text{id}$ over U_i . If no such k exists, then set $\text{TC}(X) = \infty$.

It is possible and useful to introduce a symmetrized version of topological complexity, the *symmetric topological complexity* $\text{TC}^S(X)$. This invariant appears when we restrict motion planning algorithms to be such that a motion from A to B is the reverse of a motion from B to A , [FG07].

A number of properties of topological complexity and symmetric topological complexity can be found in [Fa03, Fa06, Fa08, FG07, FG08, FY04]. The articles [FTY03, GL09] identify these concepts in the case of real projective spaces as their immersion and embedding dimensions, respectively.

The first of the two main goals of this paper is to make a thorough study of the following natural generalization of Farber’s topological complexity. As we have explained, $\mathrm{TC}(X)$ is related to the motion planning problem in which a robot with configuration space X moves from a given position to another position. More generally, we can consider a motion planning problem whose input is not only a pair of initial and final positions, but also an additional set of $n - 2$ ordered intermediate positions. Such a setting arises, for instance, in industrial production processes in which the manufacture of a given good goes through a series of production steps. The corresponding motion planning problem leads us to the homotopy invariant $\mathrm{TC}_n(X)$, the n^{th} *topological complexity* of X introduced in [Ru10], and reviewed in Section 2. Of course, the case $n = 2$ recovers Farber’s notion, except that our chosen normalization—which sets a trivial fibration to have zero Schwarz genus—gives $\mathrm{TC}(X) = \mathrm{TC}_2(X) + 1$.

In Section 3, we discuss some elementary properties of TC_n , including methods of calculation of this homotopy invariant, e.g. a relation to cup-length (Theorem 3.9) and the product inequality (Proposition 3.11). As an immediate application, the full determination of the numerical value of $\mathrm{TC}_n(X)$ is given when X is either a product of spheres (Corollary 3.12), a closed simply connected symplectic manifold (Corollary 3.15), or a quaternionic projective space (Corollary 3.16).

Many of our results generalize corresponding existing results for Farber’s TC . For instance, we show a close connection between higher topological complexity and the Lusternik-Schnirelmann category of cartesian powers of spaces.

Theorem. (Corollary 3.3) *For any path-connected space X ,*

$$\mathrm{cat}(X^{n-1}) \leq \mathrm{TC}_n(X) \leq \mathrm{cat}(X^n).$$

In fact, for a path-connected topological group G , we prove (Theorem 3.5) that $\mathrm{TC}_n(G) = \mathrm{cat}(G^{n-1})$ —this fact can be thought of as a generalization of the property $\mathrm{TC}(G) = \mathrm{cat}(G) + 1$ proved by M. Farber in [Fa04, Lemma 8.2].

In Section 4 we consider symmetric versions of higher topological complexity. We begin by introducing $\mathrm{TC}^\Sigma(X)$, a minor variation of the symmetric topological complexity $\mathrm{TC}^S(X)$ introduced by Farber and Grant in [FG07]. Although the numerical values of the two invariants differ by at most 1 (Proposition 4.4), unlike $\mathrm{TC}^S(X)$, $\mathrm{TC}^\Sigma(X)$ is a homotopy invariant (it should be noted that this property does not hold for the *monoidal topological complexity* introduced in [IS10, Definition 1.3], where the stasis property is imposed on the motion planning problem—instead of the symmetry condition we impose on TC^Σ). Indeed, we construct the corresponding higher analogues TC_n^S and TC_n^Σ , and prove the homotopy invariance of the latter (Proposition 4.11).

The calculation of the n^{th} higher *symmetric* topological complexity of a non-contractible space can turn out to be an extremely difficult task, mainly due to what seems to be poor current knowledge of precise homotopy information about braid spaces (even braid manifolds, for that matter). Our main calculations in this direction give:

Theorem. (Corollary 5.4) *For integers $k > 0$ and $n > 1$,*

$$(1) \quad \mathrm{TC}_n^S(S^k) \leq [(n+2)(k-1) + 4] (n-1)/2k.$$

At the end of Section 5 we give the following evidence toward the possible optimality of (1):

Corollary. (Corollary 5.7) The equality $\mathrm{TC}_n^S(S^k) = 2(n-1)$ holds for:

- (a) $n = 2$ and any integer $k > 0$;
- (b) $n = 3$ provided k is odd;
- (c) $n \geq 2$ provided $k = 1$.

Our second major goal, and most important theoretical achievement of this paper, is a setup of the needed technology to prove the last two results. Indeed, we need manageable combinatorial grounds for a systematic study of the homotopy properties of configuration spaces on cell complexes. For this purpose, in Section 6 we introduce a new concept: that of a *cellular stratified space*, a generalization of a cell complex where *non-closed cells* can appear. Namely, we allow cells modelled not only by a closed unit disk D^n , but by any subspace of D^n containing $\mathrm{Int}(D^n)$. Such a simple generalization of J. H. C. Whitehead's concept of a CW complex best meets the needs of the homotopy theory of open manifolds, and appears to have interesting new applications (cf. [Tn11]).

Our proposal is framed within the highly active and ongoing mutual feedback between topology and combinatorics—of which Kozlov's book [Ko08] is a good example. One of the most fundamental properties on which this relationship rests is the fact that the order complex of the face poset of a regular cell complex X is homeomorphic to X . This allows us to go back and forth between the combinatorial and topological worlds. Our Theorem 6.2.4 extends this correspondence.

Theorem. (Theorem 6.2.4) *For a regular totally normal cellular stratification \mathcal{C} on a topological space X , the classifying space (order complex) of the face poset of (X, \mathcal{C}) can be embedded in X as a strong deformation retract. Furthermore, when \mathcal{C} is a regular cell complex structure on X , the embedding coincides with the standard homeomorphism.*

Such a result provides a systematic method for constructing simplicial complexes modelling, up to homotopy, a broad class of (not necessarily compact) spaces. For example, this method can be used to obtain directly the Salvetti complex for hyperplane arrangements [Sa87], as well as its higher versions [BZ92, DS00] (see Remark 7.1.7 and Corollary 7.1.8). More important for our purposes regarding TC_n^S is the fine combinatorial control of the method, which allows us to deduce an equivariant version of the above theorem.

We demonstrate the power of the new technique by computing, in Section 7, the homotopy dimension of configuration spaces on spheres. Recall that the homotopy dimension of a space X , $\mathrm{hdim}(X)$, is the smallest dimension of a CW complex having the homotopy type of X .

Theorem. (Theorem 5.2 and Remark 5.3—proof found in Section 7) *For positive integers n and k with $n > 1$, the configuration space $B_n(S^k)$ of subsets of cardinality n in the k -dimensional sphere has*

$$(2) \quad \text{hdim}(B_n(S^k)) = (k-1)(n-1) + 1.$$

It is interesting to note that, by direct consequence of the calculations in [FZ00], the homotopy dimension of the configuration space $C_n(S^k)$ of n -tuples of distinct points in S^k is also given by the right hand side of (2) provided that $\pi_1(C_n(S^k)) = 0$. Our inspiration for addressing the case of $B_n(S^k)$ arose from an argument (based on De Concini-Salvetti's n -dimensional analogue of the Salvetti complex, [DS00]) extending the above situation to the spaces $C_n(S^2)$ which are not simply connected—i.e., those with $n \geq 3$.

Another fact to note comes from the observation that (2) recovers Kallel's upper bound (which is sharp for a number of cases) for the twisted cohomological dimension of the braid manifolds $B_n(S^k)$ (see [Ka08, Theorem 1.1]). Yet, our combinatorially-minded methods contrast with the more geometric arguments in [Ka08], in which the duality between braid spaces and truncated symmetric products plays a fundamental role. Further, P. Salvatore has informed us that the calculations in [Sa04] imply that (2) agrees with the *homology* dimension of $B_n(S^k)$ provided n is a power of 2, or k is odd and n a power of an odd prime.

There are a couple of interesting connections between (2) and recent work by Karasev and Volovikov¹: Firstly, Corollary 5.10 in [KV10] implies that, for any oriented closed k -dimensional manifold M , the inequality

$$\text{hdim}(B_n(M)) \geq (k-1)(n-1) + 1$$

holds when n is prime. We now have that such a general lower bound is optimal in view of (2). The second consequence of the interaction between (2) and Karasev-Volovikov's method is based on their concept of the *fixed point free genus*, $g_G(X)$, of a G -space X without fixed points (i.e., one where all the stabilizers are proper subgroups of G , see [KV10, Section 3]). We are interested in $G = \Sigma_n$ (for n prime) with its usual free action on $C_n(S^k)$ (recalled in the next section), and in the inequalities

$$(3) \quad g_{\mathbb{Z}_n}(C_n(S^k)) \leq \mathbf{genus}(\rho_{n,S^k}) + 1 \leq \mathbf{genus}(\pi_{n,S^k}) + 1.$$

Here $\mathbf{genus}(p)$ stands for the (normalized) Schwarz genus of the fibration p (also recalled in the next section), \mathbb{Z}_n stands for the subgroup of cyclic permutations in Σ_n , and $\rho_{n,X}: C_n(X) \rightarrow C_n(X)/\mathbb{Z}_n$ and $\pi_{n,X}: C_n(X) \rightarrow B_n(X)$ are projections onto orbit spaces. The first inequality in (3) is elementary and, in fact, an equality; the second inequality in (3) follows from the freeness of the action and the obvious existence of a (non-canonical) \mathbb{Z}_n -equivariant map $\Sigma_n \rightarrow \mathbb{Z}_n$. The point to note then is that the second inequality in (3) is also an equality. Indeed, while $(k-1)(n-1) + 2$ is a lower bound for the left-most term

¹We thank Roman Karasev for bringing this point to our attention, and for clarifying the relevant details.

in (3) ([KV10, Corollary 5.10]), it is also an upper bound for the right-most term—in view of [Sv66, Theorem 5, page 75] and (2).

Remark. Farber’s TC work was motivated in part by Smale’s ideas on the topological complexity of algorithms for finding approximations to the zeros of a complex polynomial ([Sm87]). In Smale’s view (for $m = 2$), the Schwarz genus of $\pi_{n,\mathbb{R}^m}: C_n(\mathbb{R}^m) \rightarrow B_n(\mathbb{R}^m)$ plays a fundamental role. A reasonable initial hold on the properties of this Σ_n -cover can be obtained from a good understanding of the homotopy properties of $B_n(\mathbb{R}^m)$. As done in [Ro08] (or see alternatively the speculative argument suggested in the paragraph following this remark), such a task can be accomplished by using the Fuchs-Vassiliev CW complex structure on $B_n(\mathbb{R}^m)_\infty$, the one-point compactification of $B_n(\mathbb{R}^m)$ ([Fu70, Va88]). However, a cleaner approach comes from the generalizations by Björner-Ziegler and De Concini-Salvetti in [BZ92, DS00] of the Salvetti complex, which yield a CW complex of the lowest possible dimension modeling $B_n(\mathbb{R}^m)$ up to homotopy. Example 7.1.12 in the final section of the paper reviews the construction, mainly in preparation for the situation of configuration spaces on spheres.

For the reader’s amusement, we close this introductory section with the following short speculative argument suggesting a direct way of deducing a CW complex of the smallest possible dimension modelling $B_n(\mathbb{R}^m)$ up to homotopy: Vassiliev’s work gives us a CW complex decomposition of $B_n(\mathbb{R}^m)_\infty$ for which (a) the added point at infinity is the only 0-cell, and (b) all other cells appear in dimensions in between $n + m - 1$ and nm . Thus, if the induced cellular stratified space structure on $B_n(\mathbb{R}^m)$ were regular and totally normal, then Theorem 6.2.4 would immediately yield a simplicial complex of dimension $nm - (n + m - 1) = (n - 1)(m - 1)$ embedded in $B_n(\mathbb{R}^m)$ as a strong deformation retract—an optimal result since, as recalled in (34), the homotopy dimension of $B_n(\mathbb{R}^m)$ is $(n - 1)(m - 1)$. We hope to give full support to the above reasoning by proving, in a future work, a more general form of Theorem 6.2.4.

Acknowledgments. The first author wishes to thank Michael Farber, Jesús González, Dirk Schütz and the Mathematisches Forschungsinstitut Oberwolfach for organizing a wonderful Arbeitsgemeinschaft mit aktuellem Thema in Topological Robotics. The third author is grateful for the support during a visit at the Max-Planck Institute for Mathematics in Bonn, Germany. The fourth author would like to thank the Centro di Ricerca Matematica Ennio De Giorgi, Scuola Normale Superiore di Pisa, for supporting his participation in the research program “Configuration Spaces: Geometry, Combinatorics and Topology”, during which a part of his work on this paper was done. The second, third and fourth authors were partially supported, respectively, by Conacyt Research Grant 102783, Simons Foundation Grant 209424, and Grants-in-Aid for Scientific Research, Ministry of Education, Culture, Sports, Science and Technology, Japan: 23540082. The authors wish to express their most sincere gratitude to Peter Landweber for valuable suggestions on earlier versions of this paper, and for pointing out an important extension of the authors’ original evidence for the optimality of Corollary 5.4. Preliminary portions of this work

were first put together using the system *Google wave*, a conveniently useful collaborative tool by Google Inc.

2. PRELIMINARIES

We recall Schwarz's concept of the (normalized) genus of a map [Sv66]. This is first defined below for fibrations, and then extended to arbitrary maps via fibrational substitutes.

Definition 2.1. The *Schwarz genus* (also known as *sectional category*) of a fibration $p : E \rightarrow B$ is the least number k such that there is an open covering U_0, U_1, \dots, U_k of B for which the restriction of p over each U_i , $i = 0, 1, \dots, k$, admits a continuous section.

Remark 2.2. In the situation of Definition 2.1, Schwarz's original definition in [Sv66] endows B with a genus equal to $k + 1$, that is, 1 greater than our genus. We have chosen genus k for a covering with $k + 1$ open sets to simplify our formulae, and to comply with what seems to be the most common definition of $\text{cat}(X)$, the Lusternik-Schnirelmann category (called simply 'category' for short) of X , as given in [CLOT03].

Definition 2.3. A *fibrational substitute* of a map f is a fibration \hat{f} such that there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & E \\ f \downarrow & & \downarrow \hat{f} \\ Y & \xlongequal{\quad} & Y \end{array}$$

where h is a homotopy equivalence.

Definition 2.4. The *Schwarz genus* of a map f , denoted by $\mathbf{genus}(f)$, is defined to be the Schwarz genus of its fibrational substitute. We agree to set $\mathbf{genus}(f) = -1$ for $f : X \rightarrow Y$ with $X = \emptyset = Y$.

Remark 2.5. The Schwarz genus of a map is well defined since, for a path-connected space Y , every map $f : X \rightarrow Y$ has a fibrational substitute unique up to fiber homotopy equivalence, [Do63, Se51]. Equivalently, the Schwarz genus of a map $f : X \rightarrow Y$ can be defined in terms of local homotopy sections, i.e. pairs (U, s) where U is an open subset of Y and $s : U \rightarrow X$ is such that $f \circ s$ is homotopic to the inclusion $U \hookrightarrow Y$, see [BG61].

The following result, proved in [Sv66, Proposition 22 on page 84] (see also the comments in Section 1 on page 54 of [Sv66]), will be used in the proof of Proposition 3.11. Here we agree that a normal space is, by definition, required to be Hausdorff. This convention will also be in force throughout Section 3.

Proposition 2.6. Let $f : X \rightarrow Y$ and $f' : X' \rightarrow Y'$ be two maps, and let $f \times f' : X \times X' \rightarrow Y \times Y'$ be the product map. If $Y \times Y'$ is normal, then $\mathbf{genus}(f \times f') \leq \mathbf{genus}(f) + \mathbf{genus}(f')$.

Definition 2.7. Let X be a path-connected space. The n^{th} *topological complexity* of X , $\text{TC}_n(X)$, is the Schwarz genus of the fibration

$$(4) \quad e_n^X = e_n : X^{J_n} \rightarrow X^n, \quad e_n(\gamma) = (\gamma(1_1), \dots, \gamma(1_n))$$

where J_n is the wedge of n closed intervals $[0, 1]$ (each with $0 \in [0, 1]$ as the base point), and 1_i stands for 1 in the i^{th} interval.

As explained in [Ru10], the n^{th} topological complexity is directly related to the robot motion planning problem in which the robot passes through n ordered points for $n \geq 2$. Farber's TC is just $\text{TC}_2 + 1$.

Throughout the paper, we denote by $d_n = d_n^X : X \rightarrow X^n$ the diagonal map. We can define TC_n equivalently as follows, see [Ru10, Remark 3.2.3].

Proposition 2.8. *The n^{th} topological complexity $\text{TC}_n(X)$ is equal to the Schwarz genus of the diagonal map $d_n : X \rightarrow X^n$. Indeed, e_n is a fibrational substitute of d_n .*

We close this section with some auxiliary notation relevant for the construction of our two symmetric versions of higher topological complexity.

As indicated in the introduction in the case of spheres, $C_n(X)$ stands for the configuration space of n ordered distinct points in a space X —with the subspace topology inherited from X^n . The symmetric group Σ_n acts on $e_n^{-1}(C_n(X))$ and $C_n(X)$ by permuting paths in the former case, and by permuting coordinates in the latter. These actions are free and the restricted fibration $e_n : e_n^{-1}(C_n(X)) \rightarrow C_n(X)$ is equivariant.

There is a resulting fibration $\varepsilon_n^X = \varepsilon_n : Y_n(X) \rightarrow B_n(X)$ at the level of orbit spaces, where $Y_n(X) = e_n^{-1}(C_n(X))/\Sigma_n$ and $B_n(X) = C_n(X)/\Sigma_n$ (the latter stands for the “braid” configuration space of n unordered distinct points in X , as it was defined for spheres in the introduction). Note that **genus** (ε_n^X) gives a measure of the topological complexity of the motion planning problem on X when not only a pair of end positions are relevant as input, but where $n - 2$ intermediate stages are also to be attained through the course of the motion. This concept will be connected to symmetrized forms of TC_n in Section 4. Section 5 will be devoted to exploring **genus** $(\varepsilon_n^{S^k})$.

Note that the commutative diagram (where horizontal arrows are canonical projections)

$$(5) \quad \begin{array}{ccc} e_n^{-1}(C_n(X)) & \longrightarrow & Y_n(X) \\ e_n \downarrow & & \downarrow \varepsilon_n \\ C_n(X) & \longrightarrow & B_n(X) \end{array}$$

is a pull-back square, and that the pull-back of any section of ε_n is a Σ_n -equivariant section of e_n . In particular, the homotopy fiber of ε_n is $(\Omega X)^{n-1}$, just as for e_n ([Ru10, Remark 3.2.3]). Indeed, both e_n and ε_n are fibrations all of whose fibers are homotopy equivalent to $(\Omega X)^{n-1}$. For instance, in terms of the notation introduced in the proof of Proposition 3.1 below, a copy of $(\Omega X)^{n-1}$ sits inside the fiber of e_n over an n -tuple (x_1, x_2, \dots, x_n) as the strong deformation retract consisting of multipaths $\{\gamma_j\}_{j=1}^n$ for which γ_1 is the constant path at x_1 .

3. PROPERTIES OF HIGHER TOPOLOGICAL COMPLEXITY

The higher topological complexities of a space X are closely related to the category of cartesian powers of X . The first indication of such a property comes from the inequality

$$(6) \quad \text{TC}_n(X) \leq \text{cat}(X^n)$$

which is an immediate consequence of the fact (proved in [CLOT03, Proposition 9.14]) that the Schwarz genus of a fibration does not exceed the category of the base space. On the other hand, the inequality $\text{cat}(X) \leq \text{TC}_2(X)$ is well known, and can be generalized to:

Proposition 3.1. *For any path-connected space X ,*

$$\text{cat}(X^{n-1}) \leq \text{TC}_n(X).$$

Proof. Let $\text{TC}_n(X) = k$ and choose a covering $B_0 \cup B_1 \cup \dots \cup B_k = X^n$ such that there is a continuous section s_i for e_n^X over B_i for $i = 0, \dots, k$. Let $p : X^n \rightarrow X$ be the projection onto the first factor, choose $x_1 \in X$, and put $A_i = p^{-1}(x_1) \cap B_i$.

Note that $\{A_i\}_{i=0}^k$ is an open cover for $p^{-1}(x_1)$. Since $p^{-1}(x_1)$ is homeomorphic to X^{n-1} , it suffices to show that each A_i is contractible within $p^{-1}(x_1)$.

For a point $(x_1, x_2, \dots, x_n) \in A_i$ consider the n paths $\gamma_1, \dots, \gamma_n$ in X , where γ_j is the restriction of $s_i(x_1, x_2, \dots, x_n)$ to the j -th wedge summand of J_n , for $j = 1, \dots, n$. [From now on we will express this situation by saying that $s_i(x_1, x_2, \dots, x_n)$ is the *multipath* $\{\gamma_j\}_{j=1}^n$]. So $\gamma_j(1) = x_j$ and $\gamma_j(0) = x_0$ for some $x_0 \in X$ which is independent of $j \in \{1, \dots, n\}$. Then, the constant path δ_1 at x_1 , and the paths δ_j ($j = 2, \dots, n$)—formed by using the time reversed path γ_j^{-1} the first half of the time, and γ_1 the second half—are the components of a path $\delta = (\delta_1, \dots, \delta_n)$ in $p^{-1}(x_1)$ from $\delta(0) = (x_1, x_2, \dots, x_n)$ to $\delta(1) = (x_1, x_1, \dots, x_1)$. The continuity of s_i implies that δ depends continuously on (x_1, x_2, \dots, x_n) , so we have constructed a contraction of A_i to (x_1, x_1, \dots, x_1) in $p^{-1}(x_1)$. Thus, $\text{cat}(X^{n-1}) \leq k = \text{TC}_n(X)$. \square

Remark 3.2. Using the fact that $\text{cat}(X^n) \geq n$ if X is not contractible ([CLOT03, Theorem 1.47]), we see that Proposition 3.1 recovers [Ru10, Proposition 3.5].

Proposition 3.1 and (6) yield:

Corollary 3.3. *For any path-connected space X ,*

$$\text{cat}(X^{n-1}) \leq \text{TC}_n(X) \leq \text{cat}(X^n).$$

Our next goal is to give a complete characterization of $\text{TC}_n(G)$ in terms of $\text{cat}(G^{n-1})$ for G a path-connected topological group.

Proposition 3.4. *For any path-connected topological group G ,*

$$\text{TC}_n(G) \leq \text{cat}(G^{n-1}).$$

Proof. Let ϵ denote the neutral element of G . Let $k = \text{cat}(G^{n-1})$ and choose an open covering $A_0 \cup \dots \cup A_k = G^{n-1}$ where each A_i ($i \in \{0, \dots, k\}$) contracts in G^{n-1} to an $(n-1)$ -tuple p_i . Since G is path-connected, each contracting homotopy can be extended as to arrange that $p_i = (\epsilon, \dots, \epsilon) = \epsilon^{(n-1)}$ for all $i = 0, \dots, k$.

Then, for $i \in \{0, \dots, k\}$ set

$$B_i = \{(g, ga_2, \dots, ga_n) \mid (a_2, \dots, a_n) \in A_i, g \in G\},$$

which is open in G^n . We assert that e_n^G admits a (continuous) section over each B_i . Indeed, for each i the contractibility of A_i in G^{n-1} yields a path γ_a in G^{n-1} joining $\epsilon^{(n-1)}$ to each $a = (a_2, \dots, a_n) \in A_i \subset G^{n-1}$ and depending continuously on $a \in A_i$. Augment γ_a to a path γ'_a from $\epsilon^{(n)}$ to $(\epsilon, a_2, \dots, a_n) \in B_i$ with the first coordinate remaining constant. Then, for any $g \in G$, $g\gamma'_a$ is a path joining $(g, \dots, g) = g\epsilon^{(n)} \in G^n$ to $(g, ga_2, \dots, ga_n) \in B_i$ and depending continuously on n -tuples in B_i . Then, we get the required section

$$s_i : B_i \rightarrow G^{J_n}$$

where, on the j^{th} interval of J_n , $s_i(g, ga_2, \dots, ga_n)$ is the j^{th} coordinate of $g\gamma'_a$.

The proof will be complete once we check that $B_0 \cup \dots \cup B_k = G^n$. Take $(b_1, \dots, b_n) \in G^n$ and put $g = b_1$ and $a_i = g^{-1}b_i$. Then there exists j such that $(a_2, \dots, a_n) \in A_j$. So, $(b_1, \dots, b_n) \in B_j$. \square

Corollary 3.3 and Proposition 3.4 combined yield:

Theorem 3.5. *For any path-connected topological group G ,*

$$\text{TC}_n(G) = \text{cat}(G^{n-1}).$$

Alternatively, we can look at the growth of TC_n in terms of the difference of any two consecutive values of n .

Corollary 3.6. *Let G be a path-connected topological group all of whose finite cartesian powers G^k are normal². Then for $n \geq 3$,*

$$\text{TC}_n(G) - \text{TC}_{n-1}(G) \leq \text{cat}(G).$$

Proof. This is a consequence of Theorem 3.5 and the product inequality for the category—valid under the current normality assumptions, in view of Proposition 2.6. \square

Unlike with topological groups, higher topological complexities of an arbitrary path-connected space X do not appear to be completely determined by the category of cartesian powers of X . Nonetheless, we can directly obtain the following bound on the difference of two consecutive higher topological complexities of X .

²As noted in Section 2, we assume that a normal space is, by definition, Hausdorff. Thus, in view of the classical Birkhoff-Kakutani theorem, the normality hypothesis in Corollary 3.6 holds when G satisfies the first axiom of countability—i.e. provided G is metrizable.

Proposition 3.7. *Let X be a path-connected space all of whose finite cartesian powers X^k are normal. Then for $n \geq 3$,*

$$\mathrm{TC}_n(X) - \mathrm{TC}_{n-1}(X) \leq \mathrm{cat}(X^2).$$

Proof. Use the argument in the proof of Corollary 3.6, replacing Theorem 3.5 by the inequalities in Corollary 3.3. \square

In particular $\mathrm{TC}_n(X)$ is bounded from above by a linear function on n with slope $\mathrm{cat}(X^2)$. However, this is quite coarse since, according to [Ru10, (5.1)], $\mathrm{TC}_n(X) \leq n \mathrm{TC}_2(X)$.

Next we consider the higher analogue of the usual cup-length lower bound for TC. In the following definition (modified cup-length of a space X), we consider cohomology with local coefficients.

Definition 3.8. Given a space X and a positive integer n , define the d_n -cup-length, denoted by $\mathrm{cl}(X, n)$, to be the largest integer m with the following property: There exist cohomology classes $u_i \in H^*(X^n; A_i)$ such that $d_n^* u_i = 0$ for $i = 1, \dots, m$ and

$$u_1 \smile \dots \smile u_m \neq 0 \in H^*(X^n; A_1 \otimes \dots \otimes A_m).$$

The following theorem, which follows directly from [Sv66, Theorem 4], gives a lower bound for TC_n in terms of $\mathrm{cl}(X, n)$.

Theorem 3.9. *For any path-connected space X we have the inequality $\mathrm{cl}(X, n) \leq \mathrm{TC}_n(X)$.*

We will also need the following bound on $\mathrm{cl}(X \times S^k, n)$ in terms of $\mathrm{cl}(X, n)$.

Theorem 3.10. *For any path-connected space X and positive integers n and k we have $\mathrm{cl}(X \times S^k, n) \geq \mathrm{cl}(X, n) + n - 1$. This inequality can be improved to $\mathrm{cl}(X \times S^k, n) \geq \mathrm{cl}(X, n) + n$ provided k is even and $H^*(X)$ is torsion-free.*

Proof. Let v be a generator of $H^k(S^k) = \mathbb{Z}$. Let $p_i : (S^k)^n \rightarrow S^k$ be the projection onto the i^{th} factor and put $v_i = p_i^*(v)$ for $i = 1, \dots, n$. Assume that $\mathrm{cl}(X, n) = m$ and take u_1, \dots, u_m such that $d_n^*(u_j) = 0$ for $j = 1, \dots, m$ and $u_1 \smile \dots \smile u_m \neq 0$.

To prove the first assertion note that $d_n^*(v_i - v_1) = 0$ for $i > 1$, while the basis element $v_2 \smile \dots \smile v_n \in H^*((S^k)^n)$ appears in the reduced expansion (using distributivity) of $(v_2 - v_1) \smile \dots \smile (v_n - v_1)$. Hence,

$$u_1 \smile \dots \smile u_m \smile (v_2 - v_1) \smile \dots \smile (v_n - v_1) \neq 0.$$

Thus $\mathrm{cl}(X \times S^k, n) \geq \mathrm{cl}(X, n) + n - 1$.

Assume now that k is even and that $H^*(X)$ is torsion-free. The element $v_1 + v_2 + \dots + v_{n-1} - (n-1)v_n$ lies in the kernel of d_n^* and has cup n^{th} power equal to a non-zero multiple of $v_1 \smile v_2 \smile \dots \smile v_n$. Hence,

$$u_1 \smile \dots \smile u_m \smile (v_1 + v_2 + \dots + v_{n-1} - (n-1)v_n)^n \neq 0.$$

Thus $\mathrm{cl}(X \times S^k, n) \geq \mathrm{cl}(X, n) + n$. \square

In [Fa03] M. Farber showed that, with suitable topological hypothesis, $\mathrm{TC}(X \times Y) \leq \mathrm{TC}(X) + \mathrm{TC}(Y) - 1$, that is, $\mathrm{TC}_2(X \times Y) \leq \mathrm{TC}_2(X) + \mathrm{TC}_2(Y)$. This result is generalized in the following proposition.

Proposition 3.11. *Let X and Y be path-connected spaces. If $(X \times Y)^n$ is normal, then $\mathrm{TC}_n(X \times Y) \leq \mathrm{TC}_n(X) + \mathrm{TC}_n(Y)$.*

Proof. The natural homeomorphisms

$$(X \times Y)^n \rightarrow X^n \times Y^n,$$

$$((x_1, y_1), \dots, (x_n, y_n)) \mapsto (x_1, \dots, x_n, y_1, \dots, y_n), \quad x_i \in X, \quad y_j \in Y$$

and

$$(X \times Y)^{J_n} \rightarrow X^{J_n} \times Y^{J_n},$$

$$(\varphi : J_n \rightarrow X \times Y) \mapsto ((p_X \circ \varphi : J_n \rightarrow X), (p_Y \circ \varphi : J_n \rightarrow Y))$$

fit into the commutative diagram

$$\begin{array}{ccc} (X \times Y)^{J_n} & \longrightarrow & X^{J_n} \times Y^{J_n} \\ e_n^{X \times Y} \downarrow & & \downarrow e_n^X \times e_n^Y \\ (X \times Y)^n & \longrightarrow & X^n \times Y^n. \end{array}$$

So, the desired conclusion follows directly from Proposition 2.6. \square

Next, we apply the previous results to compute the higher topological complexities of concrete families of spaces.

Corollary 3.12. $\mathrm{TC}_n(S^{k_1} \times S^{k_2} \times \dots \times S^{k_m}) = m(n-1) + l$ where l is the number of even dimensional spheres.

Proof. Note that $\mathrm{TC}_n(S^k) = \mathrm{cl}(S^k, n)$ for all k , [Ru10, Section 4]. Then the inequality $\mathrm{cl}(S^{k_1} \times \dots \times S^{k_m}, n) \geq m(n-1) + l$ follows from Theorem 3.10 by induction, so $\mathrm{TC}_n(S^{k_1} \times \dots \times S^{k_m}) \geq m(n-1) + l$ by Theorem 3.9. The opposite estimate follows from Proposition 3.11. \square

The calculation of the n^{th} topological complexity of the k -dimensional torus $T^k = (S^1)^k$, partially solved for $k = 2$ in [Ru10, Proposition 5.1], can now be completed using either Corollary 3.12 or Theorem 3.5.

Corollary 3.13. $\mathrm{TC}_n(T^k) = k(n-1)$.

Theorem 3.14. *Let X be a CW complex of finite type, and R a principal ideal domain. Take $u \in H^d(X; R)$ with $d > 0$, d even, and assume that the n -fold iterated self R -tensor product $u^m \otimes \dots \otimes u^m \in (H^{md}(X; R))^{\otimes n}$ is an element of infinite additive order. Then $\mathrm{TC}_n(X) \geq mn$.*

Proof. For $i = 1, \dots, n$, let $p_i : X^n \rightarrow X$ be the projection onto the i^{th} factor and put $u_i = p_i^*(u) \in H^d(X^n; R)$. In view of Theorem 3.9, the required inequality follows from

$$(7) \quad v := (u_2 - u_1)^{2m}(u_3 - u_1)^m \cdots (u_n - u_1)^m \neq 0.$$

In order to check (7), note that v comes from the tensor product, which injects into the cohomology of the cartesian product by the Künneth Theorem (this is where the finiteness hypotheses are used). So, calculations can be performed in the former R -module. Now, assuming that $\dim(X) \leq dm + 1$, we have

$$\begin{aligned} v &= (u_2 - u_1)^{2m}(u_3 - u_1)^m \cdots (u_n - u_1)^m \\ &= (-1)^m \binom{2m}{m} u_1^m u_2^m (u_3 - u_1)^m \cdots (u_n - u_1)^m \\ &= (-1)^m \binom{2m}{m} u_1^m u_2^m u_3^m (u_4 - u_1)^m \cdots (u_n - u_1)^m \\ &= \cdots \\ &= (-1)^m \binom{2m}{m} u_1^m u_2^m \cdots u_n^m, \end{aligned}$$

which is non-zero by hypothesis. On the other hand, for $\dim(X)$ arbitrary, consider the skeletal inclusion $j : X^{(dm+1)} \rightarrow X$ and note that $v \neq 0$ since $j^*(v) \neq 0$. \square

Corollary 3.15. *For every closed simply connected symplectic manifold M^{2m} we have $\text{TC}_n(M) = nm$.*

Proof. This follows from Theorem 3.14 (taking u to be the cohomology class given by the symplectic 2-form on M , and noting that the hypothesis on $u^m \otimes \cdots \otimes u^m$ holds since the coefficients are taken over the reals), inequality (6), the product inequality for category, and the inequality $\text{cat}(M^{2m}) \leq m$ which follows from [Sv66, Theorem 5, page 75]. (The argument derived from this even yields $\text{cat}(M^{2m}) = m$, a fact that is well known to experts.) \square

Of course, Corollary 3.15 applies to complex projective spaces. In the quaternionic case essentially the same proof gives:

Corollary 3.16. *The quaternionic projective space of real dimension $4m$, $\mathbb{H}\mathbb{P}^m$, has $\text{TC}_n(\mathbb{H}\mathbb{P}^m) = nm$.*

In the next section we introduce two symmetric versions of TC_n . One of them, $\text{TC}_n^\Sigma(X)$, has the advantage of being a homotopy invariant of X . The other, $\text{TC}_n^S(X)$, gives (up to the normalization convention in Remark 2.2) the natural generalization of the symmetric topological complexity $\text{TC}^S(X)$ introduced by Farber and Grant in [FG07]. Section 5 represents our contribution toward computing $\text{TC}_n^S(X)$ when X is a sphere.

4. SYMMETRIC TOPOLOGICAL COMPLEXITY

Farber and Grant studied in [FG07] a symmetrized version of topological complexity TC^S . Here we begin by proposing a homotopically more suitable version TC^Σ . Although the homotopy invariant TC^Σ is inspired by [FG08, Definition 2], those authors did not develop this approach.

Consider the involutions $\tau : X \times X \rightarrow X \times X$ and $\bar{\tau} : PX \rightarrow PX$ defined by $\tau(x, y) = (y, x)$ and $\bar{\tau}(\gamma)(t) = \gamma(1 - t)$, for $(x, y) \in X \times X$ and $\gamma \in PX$.

Definition 4.1. A subset A in $X \times X$ is *symmetric* if $\tau A = A$.

Definition 4.2. A function $s : A \rightarrow PX$ is *equivariant* if $\bar{\tau}(s(a)) = s(\tau(a))$ for $a \in A$, where A is a symmetric subset of $X \times X$.

Definition 4.3. $\mathrm{TC}^\Sigma(X)$ is the least number k such that $X \times X = A_0 \cup A_1 \cup \cdots \cup A_k$ where each A_i is open, symmetric, and has a continuous equivariant section $s_i : A_i \rightarrow PX$ of the map e_2 in (4).

Before proving (in Proposition 4.11 below) that $\mathrm{TC}^\Sigma(X)$ is a homotopy invariant of X , we show that its numerical value differs by at most 1 from the numerical value of $\mathrm{TC}^S(X)$. In our terms the Farber-Grant definition amounts to setting

$$\mathrm{TC}^S(X) = 2 + \mathbf{genus}(\varepsilon_2)$$

where ε_2 is the map on the right hand side of (5). However, in accordance with the normalization discussed in the introductory section (implicit in Definition 4.3), we should compare $\mathrm{TC}^\Sigma(X)$ with

$$(8) \quad \mathrm{TC}_2^S(X) = 1 + \mathbf{genus}(\varepsilon_2).$$

Proposition 4.4. *For each ENR X we have*

$$\mathrm{TC}_2^S(X) - 1 \leq \mathrm{TC}^\Sigma(X) \leq \mathrm{TC}_2^S(X).$$

Remark 4.5. We will prove a more general version of Proposition 4.4 (Theorem 4.12 below). The proof of the general version is considerably more elaborate as it requires an involved use of the theory of *equivariant* euclidean neighborhood retracts. For the sake of clarity, we offer first the easy argument proving Proposition 4.4, which will also serve as a warm-up for the proof of Theorem 4.12.

Proof of Proposition 4.4. To prove the first inequality, take an open covering $X \times X = A_0 \cup \cdots \cup A_k$ where each A_i is symmetric and has a continuous equivariant section of e_2 . The $\mathbb{Z}/2$ -action τ on $X \times X$ yields the orbit map $\rho_2 : X \times X \rightarrow (X \times X)/\tau$. Then, for each $i = 0, \dots, k$, $\rho_2(A_i - d_2(X))$ is open and has a section of ε_2 , and thus $\mathbf{genus}(\varepsilon_2) \leq \mathrm{TC}^\Sigma(X)$. For the second inequality, take B_0, \dots, B_l , with $B_0 \cup \cdots \cup B_l = \rho_2(X \times X - d_2(X))$ where each B_i is open and has a section of ε_2 . Then each $\rho_2^{-1}(B_i)$ is symmetric, open in $X \times X$, and admits an equivariant section of e_2 , cf. [FG07, Lemma 8]. Further, since X is an ENR, there is a symmetric open neighborhood of $d_2(X)$ supporting an equivariant section of e_2 (see the proof of [FG07, Corollary 9]). Consequently $\mathrm{TC}^\Sigma(X) \leq 1 + \mathbf{genus}(\varepsilon_2)$. \square

As the following examples show, Proposition 4.4 is optimal in the sense that the two bounds given in this result are attained.

Example 4.6. For X contractible we have $\mathrm{TC}_2(X) = \mathrm{TC}^\Sigma(X) = 0$ while $\mathrm{TC}_2^S(X) = 1$. Indeed, take a point $x_0 \in X$ and a contraction $H : X \times I \rightarrow X$, with $H(x, 0) = x$ and $H(x, 1) = x_0$ for all $x \in X$. Given $(a, b) \in X \times X$, take the path $\sigma = s(a, b) : I \rightarrow X$ such that $\sigma(t) = H(a, 2t)$ for $0 \leq t \leq 1/2$ and $\sigma(t) = H(b, 2 - 2t)$ for $1/2 \leq t \leq 1$. Then s is an equivariant section for e_2^X and, in view of the general inequality

$$\mathrm{TC}_2(X) \leq \mathrm{TC}^\Sigma(X),$$

this gives $\mathrm{TC}_2(X) = \mathrm{TC}^\Sigma(X) = 0$. The same argument, but now using (8), gives $\mathrm{TC}_2^S(X) = 1$ (see [FG08, Example 7]).

Example 4.7. Farber and Grant proved in [FG07, Corollary 18] that $\mathrm{TC}_2^S(S^k) = 2$ for any k . On the other hand, Farber proved in [Fa03] that $\mathrm{TC}_2(S^k) = 1$ for k odd, while $\mathrm{TC}_2(S^k) = 2$ for k even. Here we observe that

$$(9) \quad \mathrm{TC}_2(S^k) = \mathrm{TC}^\Sigma(S^k) = \mathrm{TC}_2^S(S^k) \text{ if } k \text{ is even,}$$

for $2 = \mathrm{TC}_2(S^k) \leq \mathrm{TC}^\Sigma(S^k) \leq \mathrm{TC}_2^S(S^k) = 2$. For k odd the construction from [Fa08, Example 4.8] gives an open covering $S^k \times S^k = A_0 \cup A_1$ by symmetric sets A_i , and continuous sections of e_2 over each A_i , $i = 0, 1$. However, one of these sections is not equivariant, which prevents us from deducing $\mathrm{TC}^\Sigma(S^k) = 1$.

We next generalize Definitions 4.1 and 4.2 to obtain higher analogues of TC^Σ . Recall that for a given n , the symmetric group Σ_n acts on X^n and on X^{J_n} by permuting coordinates and paths respectively. Further, the fibration e_n in (4) is Σ_n -equivariant.

Definition 4.8. A subset A in X^n is *symmetric* if $\sigma A = A$ for all $\sigma \in \Sigma_n$.

Definition 4.9. For a symmetric $A \subset X^n$, a function $s : A \rightarrow X^{J_n}$ is *equivariant* if $\sigma(s(a)) = s(\sigma(a))$ for all $a \in A$ and $\sigma \in \Sigma_n$.

Definition 4.3 can now be extended to:

Definition 4.10. $\mathrm{TC}_n^\Sigma(X)$ is the least number k such that $X^n = A_0 \cup A_1 \cup \dots \cup A_k$ where each A_i is open, symmetric and has a continuous equivariant section $s_i : A_i \rightarrow X^{J_n}$ for e_n .

So, $\mathrm{TC}_2^\Sigma(X) = \mathrm{TC}^\Sigma(X)$.

Proposition 4.11. $\mathrm{TC}_n^\Sigma(X)$ is a homotopy invariant of X .

Proof. It suffices to prove that, given $f : Y \rightarrow X$ and $g : X \rightarrow Y$ with $gf \simeq 1_Y$, we have $\mathrm{TC}_n^\Sigma(X) \geq \mathrm{TC}_n^\Sigma(Y)$ for all n . Let $H : 1_Y \simeq gf$ be a homotopy $H : Y \times [0, 1] \rightarrow Y$ such that $H(y, 0) = y$ and $H(y, 1) = gf(y)$.

Let A be an open symmetric subset of X^n , and let $s : A \rightarrow X^{J_n}$ be an equivariant section of e_n^X over A . Given $a = (a_1, \dots, a_n) \in A$, let $s_i(a)$ denote the restriction of $s(a) \in X^{J_n}$ to

the i^{th} wedge summand of J_n (this is a path in X joining x_0 and a_i for some $x_0 \in X$ that depends continuously on a). Note that the equivariance of s gives

$$(10) \quad s_i(a_{\sigma(1)}, \dots, a_{\sigma(n)}) = s_{\sigma(i)}(a_1, \dots, a_n) \quad \text{for } \sigma \in \Sigma_n.$$

Take $B := (f^n)^{-1}(A)$ and consider the map $s' : B \rightarrow Y^{J_n}$ which, at a given $b \in B$ with $f^n(b) = a$, has $s'_i(b) := (g \circ s_i(a)) \cdot \gamma_i$ as its restriction to the i^{th} wedge summand of J_n , where γ_i is the path in Y given by

$$\gamma_i(t) = H(b_i, 1 - t).$$

Then, s' is an equivariant continuous section of e_n^Y over B (the equivariance of s' follows from (10)).

Now, if $X = A_0 \cup \dots \cup A_k$ where each A_j ($j = 0, \dots, k$) is open, symmetric, and admits a continuous equivariant section of e_n^X , then $Y = B_0 \cup \dots \cup B_k$ where each B_j —defined as above using A_j —is open, symmetric, and admits a continuous equivariant section of e_n^Y . Hence, $\text{TC}_n^\Sigma(X) \geq \text{TC}_n^\Sigma(Y)$. \square

The following assertion is our higher analogue of Proposition 4.4.

Theorem 4.12. *If X is an ENR, and ε_n is the map on the right hand side of (5), then*

$$(11) \quad \text{genus}(\varepsilon_n) \leq \text{TC}_n^\Sigma(X) \leq \text{genus}(\varepsilon_n) + \dots + \text{genus}(\varepsilon_2) + n - 1.$$

The first inequality in (11) follows just as in the proof of Proposition 4.4: If e_n admits an equivariant section over $A \subset X^n$, then ε_n admits a section over $\rho_n(A \cap C_n(X))$ where $\rho_n : X^n \rightarrow X^n/\Sigma_n$ stands for the canonical projection. Our efforts will therefore focus on the second inequality in (11), whose proof requires some preparation.

Definition 4.13. A topological space X with an action of a compact Lie group G is called a *euclidean neighborhood G -retract* (G -ENR) if X can be G -equivariantly embedded, as a G -equivariant retract of a G -symmetric neighborhood of X , into an orthogonal representation of G .

In what follows we will make implicit use of the following fact: if a G -ENR X is G -equivariantly embedded in a given orthogonal representation \mathbb{R}^N of G , then there exists a G -symmetric neighborhood U of X in \mathbb{R}^N and a G -equivariant retraction $U \rightarrow X$. As suggested at the end of the introduction in [J76], such a property follows by applying the equivariant version of the Tietze Theorem (Tietze-Gleason Theorem, [Br72, Gl50]) to the non-equivariant argument in [Do95, Proposition and Definition IV.8.5].

We shall use the following weaker version of [J76, Theorem 2.1]³.

Theorem 4.14 (Jaworowski). *Let L be a finite group acting on an ENR Z . Then Z is an L -ENR if for every subgroup G of L , the fixed point set Z^G is an ENR.*

³Although Jaworowski's theorem was originally set in terms of a combination of the concepts of ANR's and ENR's, for our formulation the reader should keep in mind the fact that any ENR is an ANR (which is elementary in view of the Tietze Theorem).

There is a Σ_n -equivariant filtration

$$d_n(X) = D^1(X) \subset \cdots \subset D^{n-1}(X) \subset D^n(X) = X^n$$

where, for $i \in \{1, \dots, n\}$, $D^i(X)$ is the closed set consisting of the n -tuples (x_1, x_2, \dots, x_n) such that the set $\{x_1, x_2, \dots, x_n\}$ has cardinality at most i . For instance, $D^{n-1}(X)$ is the fat diagonal in X^n , denoted by $\Delta_n(X)$ in Corollary 7.1.2 in our final section. Compare this filtration with the filtration considered at the end of Section 1 in [Ka08].

Set $D^0(X) = \emptyset$, and for $1 \leq i \leq n$ let C^i stand for the difference $D^i(X) - D^{i-1}(X)$, the subspace of n -tuples (x_1, x_2, \dots, x_n) such that the set $\{x_1, x_2, \dots, x_n\}$ has cardinality i . Note that $C^n = C_n(X)$ and that for $i < n$, each partition $\mathcal{P} = \{P_1, \dots, P_i\}$ of $\{1, 2, \dots, n\}$ into i nonempty sets determines a closed subspace $C_{\mathcal{P}}^i \subset C^i$, formed by those tuples (x_1, \dots, x_n) in C^i satisfying $x_r = x_s$ whenever both r and s lie in the same part P_j for some j .

Note that C^i is the disjoint union of the $C_{\mathcal{P}}^i$'s, each of which maps homeomorphically onto $C_i(X)$ under a suitable coordinate projection. [For instance, for $n = 3$ the three closed subspaces partitioning C^2 are determined by the three requirements $x_1 = x_2$, $x_1 = x_3$, and $x_2 = x_3$; in the latter case, the required projection can be chosen to be $(x_1, x_2, x_3) \mapsto (x_1, x_2)$.] Therefore, we have a continuous (surjective) map $\pi_i: C^i \rightarrow C_i(X)$.

Let P^i denote the subspace of $e_n^{-1}(C^i)$ consisting of those multipaths $\alpha = \{\alpha_i\}_{i=1}^n$ satisfying $\alpha_k = \alpha_\ell$ whenever $\alpha_k(1_k) = \alpha_\ell(1_\ell)$. Proceeding as above, we get a continuous surjection $\Pi_i: P^i \rightarrow e_i^{-1}(C_i(X))$ in such a way that in the following commutative diagram

$$(12) \quad \begin{array}{ccccccc} X^{J_n} & \longleftarrow & P^i & \xrightarrow{\Pi_i} & e_i^{-1}(C_i(X)) & \longrightarrow & Y_i(X) \\ e_n \downarrow & & e_n \downarrow & & e_i \downarrow & & \downarrow \varepsilon_i \\ X^n & \longleftarrow & C^i & \xrightarrow{\pi_i} & C_i(X) & \longrightarrow & B_i(X) \end{array}$$

the second and third squares are pullbacks, and the two left-most horizontal maps are inclusions but do not determine a pullback square.

Our last ingredient in preparation for the proof of (11) is given by taking an arbitrary open subset W of $B_i(X)$. We then let $A = \pi_i^{-1}(W')$ where W' stands for the inverse image of W under the projection $C_i(X) \rightarrow B_i(X)$. Clearly W' is Σ_i -symmetric and A is Σ_n -symmetric. This setup will be in force in the following two auxiliary results, which are the basis of our proof of the second inequality in (11).

Lemma 4.15. *The space A is a Σ_n -ENR.*

Proof. Note first that every $C_{\mathcal{P}}^i$ is an ENR, because it is homeomorphic to $C_i(X)$ which, in turn, is an open subset of the ENR X^i . Now, every $g \in \Sigma_n$ yields a homeomorphism from any given $C_{\mathcal{P}}^i$ onto some $C_{\mathcal{P}'}^i$. In particular for $\mathcal{P} = \mathcal{P}'$, if there is some $x \in C_{\mathcal{P}}^i$ fixed by g , then $g \cdot y = y$ for all $y \in C_{\mathcal{P}}^i$, i.e. $(C_{\mathcal{P}}^i)^g = C_{\mathcal{P}}^i$. Hence, for any subgroup G of Σ_n , the set $(C_{\mathcal{P}}^i)^G$ is either empty or the whole $C_{\mathcal{P}}^i$, and therefore an ENR. Consequently, $(C^i)^G$ is

an ENR since C^i is the disjoint union of the various C_P^i 's, and A^G is an ENR since A is open in C^i . Thus, by Theorem 4.14, A is a Σ_n -ENR, as asserted. \square

Lemma 4.16. *Assume $s : A \rightarrow P^i$ is a Σ_n -equivariant section of the second vertical map in (12). Then there is a Σ_n -symmetric neighborhood U of A in X^n that admits a Σ_n -equivariant section $\sigma : U \rightarrow X^{J_n}$ of the first vertical map in (12).*

Proof. We begin by noticing that, as a consequence of Theorem 4.14, X^n is a Σ_n -ENR. Indeed, for any subgroup G of Σ_n , the fixed point set of G on X^n is an intersection of hyperplanes $x_i = x_j$ in X^n . Hence, $(X^n)^G$ is an ENR since it is homeomorphic to X^m for $m \leq n$. Thus, we can take Σ_n -equivariant embeddings $A \rightarrow X^n \rightarrow \mathbb{R}^N$, and a Σ_n -equivariant retraction $r' : O \rightarrow A$ of a Σ_n -symmetric neighborhood O of A in \mathbb{R}^N , where \mathbb{R}^N is an orthogonal representation of Σ_n .

Set $V = O \cap X^n$. Then V is a Σ_n -symmetric neighborhood of A in X^n , and $r = r'|_V : V \rightarrow A$ is a Σ_n -equivariant retraction. Note that V is an open Σ_n -symmetric subset of the Σ_n -ENR X^n , and so V is a Σ_n -ENR too. We can then choose an open Σ_n -symmetric neighborhood Y of V in \mathbb{R}^N , and a Σ_n -equivariant retraction $\rho : Y \rightarrow V$. Let $U \subset V$ consist of all points $v \in V$ such that the segment from v to $i \circ r(v)$ lies in Y where i stands for the inclusion $A \hookrightarrow V$ (cf. [Do95, Corollary IV.8.7]). Clearly U is a neighborhood of A in V , and hence in X^n . Furthermore, the composition $i \circ r|_U$ and the inclusion $U \hookrightarrow V$ are homotopic via the homotopy

$$\Phi : U \times I \rightarrow V, \quad \Phi(u, t) = \rho(t \cdot u + (1 - t) \cdot i \circ r(u)).$$

Note that U is Σ_n -symmetric and Φ is Σ_n -equivariant, since the Σ_n -action on \mathbb{R}^N is orthogonal and so it maps lines to lines.

We use the homotopy Φ in order to construct a Σ_n -equivariant section $\sigma : U \rightarrow X^{J_n}$ of the first vertical map in (12). For $x \in U$, consider the path $\beta : I \rightarrow V$, $\beta(t) = \Phi(x, t)$, starting at $y = \beta(0) = r(x) \in A$ and ending at x . Since V is a subset of X^n , we can set $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, and $\beta = (\beta_1, \dots, \beta_n)$, so each β_i is a path in X from y_i to x_i . Further, $s(y)$ gives a multipath $\{\alpha_i\}_{i=1}^n$ with $\alpha_i(1) = y_i$ and $\alpha_i(0) = \alpha_j(0)$ for all $1 \leq i, j \leq n$. Then the multipath $\{\alpha_i \cdot \beta_i\}_{i=1}^n$ determines an element $\sigma(x) \in X^{J_n}$ with $e_n(\sigma(x)) = x$. This defines the required Σ_n -equivariant section over U . \square

Note that the two pull-back squares in (12) imply that the hypothesis in Lemma 4.16 holds whenever W (the arbitrary open subset of $B_i(X)$ taken in the paragraph previous to Lemma 4.15) is chosen to admit a section of the fourth vertical map in (12). Thus we obtain the following:

Proof of Theorem 4.12. In view of Lemmas 4.15 and 4.16 we can choose $1 + \mathbf{genus}(\varepsilon_i)$ Σ_n -equivariant local sections for e_n whose domains cover C^i , and thus a total of

$$(13) \quad \sum_{i=2}^n (1 + \mathbf{genus}(\varepsilon_i)) + 1 = \mathbf{genus}(\varepsilon_n) + \dots + \mathbf{genus}(\varepsilon_2) + n$$

Σ_n -equivariant local sections for e_n whose domains cover X^n , where the last “+1” in (13) accounts for the obvious equivariant section on the diagonal $D^1(X)$. The theorem follows. \square

A comparison of Proposition 4.4 and Theorem 4.12 suggests the following generalization of (8):

Definition 4.17. For $n \geq 2$ set

$$\mathrm{TC}_n^S(X) = \mathbf{genus}(\varepsilon_n) + \cdots + \mathbf{genus}(\varepsilon_2) + n - 1.$$

This variation of the one proposed in the short final section in [Ru10] will be explored next for X a sphere.

5. BOUNDING $\mathbf{genus}(\varepsilon_n)$ FOR SPHERES

The following result, an application of [Sv66, Theorem 5, page 75] and the fact that $(\Omega X)^{n-1}$ is the homotopy fiber of the map $\varepsilon_n: Y_n(X) \rightarrow B_n(X)$ in (5), is the basis for this section’s goal.

Proposition 5.1. *If X is an $(s-1)$ -connected space and $B_n(X)$ has the homotopy type of a d -dimensional CW space, then $\mathbf{genus}(\varepsilon_n) \leq d/s$.*

The following paragraph illustrates our strategy to settle a (potentially optimal) upper bound for $\mathrm{TC}_n^S(S^k)$.

As recalled in Example 4.7, the equality $\mathrm{TC}_2^S(S^k) = 2$ holds for any k . Farber and Grant prove that $\mathrm{TC}_2^S(S^k)$ is no greater than 2 by producing a symmetric motion planner with two local rules. Their construction makes use of a well-known explicit Σ_2 -equivariant deformation retraction

$$(14) \quad C_2(S^k) \rightarrow S^k$$

that implies a homotopy equivalence $B_2(S^k) \simeq \mathbb{RP}^k$. But note that Proposition 5.1 gives an alternative direct way to deduce the inequality $\mathrm{TC}_2^S(S^k) \leq 2$, for $B_2(S^k)$ is homotopy equivalent to a CW space of dimension k .

We next give the main ingredient for extending the previous argument in order to obtain a strong upper bound for $\mathrm{TC}_n^S(S^k)$.

Theorem 5.2. *Put $d(k, n) = (k-1)(n-1) + 1$. For $n \geq 2$ and $k \geq 1$, $B_n(S^k)$ has the homotopy type of a finite CW complex of dimension $d(k, n)$.*

Note that the case $k = 1$ in Theorem 5.2 is well known as

$$(15) \quad B_n(S^1) \text{ has the homotopy type of } S^1$$

(cf. [Ka08, Proposition 2.5]). Our proof of Theorem 5.2 can be thought of as a rather elaborate generalization of the case $n = 2$ —given by (14). Namely, we construct (in Theorem 7.2.5 at the end of the paper) an explicit finite simplicial complex of dimension $d(k, n)$ embedded in $C_n(S^k)$ as a strong Σ_n -equivariant deformation retract.

Remark 5.3. The full power of Theorem 5.2 and, in particular, the assertion in (2) are obtained after observing that

$$d(k, n) \geq \text{hdim}(B_n(S^k)) \geq \text{hdim}(C_n(S^k)) \geq d(k, n).$$

The first inequality is a restatement of Theorem 5.2, and the last one follows from the calculations in [FZ00]. The middle inequality is standard: the homotopy dimension of a space is not less than the homotopy dimension of any of its covering spaces. It is interesting to compare with the situation in (34) at the end of the Subsection 7.1.

An immediate consequence of Proposition 5.1 and Theorem 5.2 is:

Corollary 5.4. *For $X = S^k$ and $i \geq 2$, $\text{genus}(\varepsilon_i) \leq i - 1 - (i - 2)/k$. In particular, for $n \geq 2$,*

$$\text{TC}_n^S(S^k) \leq [(n + 2)(k - 1) + 4] (n - 1)/2k.$$

The remainder of the section is devoted to presenting evidence toward the potential optimality of Corollary 5.4. As a warm-up for the method used, we note that Corollary 5.4 is optimal for $i = n = 2$. Indeed, Farber and Grant prove in [FG07, Section 3] the inequality

$$(16) \quad \text{TC}_2^S(S^k) \geq 2.$$

This is done by means of an involved extension of Haefliger's calculation of the mod 2 cohomology ring $H^*(B_2(M); \mathbb{Z}/2)$ for M a closed smooth manifold, but a more conceptual argument is available: Start by observing that if (16) were to fail, then there would exist a continuous section σ for $\varepsilon_2^{S^k}$. In such a situation the adjoint to the composite

$$S^k \xrightarrow{\alpha} C_2(S^k) \xrightarrow{\tilde{\sigma}} e_2^{-1}(C_2(S^k)) \hookrightarrow PS^k,$$

where $\alpha(x) = (x, -x)$, and $\tilde{\sigma}$ is the $(\mathbb{Z}/2)$ -equivariant pull-back of σ under (5), would yield a homotopy $H: S^k \times [0, 1] \rightarrow S^k$ between the identity $H(-, 0)$ and the antipodal map $H(-, 1)$, and which would in addition satisfy the relation

$$(17) \quad H(x, t) = H(-x, 1 - t).$$

But this is impossible since the identity on S^k (which has degree 1) cannot be homotopic to the presumed map $H(-, 1/2)$ which, in view of (17), would factor as $S^k \xrightarrow{\text{proj}} \mathbb{RP}^k \rightarrow S^k$, and would therefore have even degree.

The above argument, as well as the—closely related—proof of Proposition 5.5 below, were pointed out to the authors by Peter Landweber. Together with Corollary 5.7 at the end of the section, this gives the promised evidence toward the optimality of Corollary 5.4.

Proposition 5.5. *Let k be a positive odd integer. For $X = S^k$ and $i \geq 2$, $\text{genus}(\varepsilon_i) > 0$. Further, $\text{genus}(\varepsilon_i) = 1$ provided $i = 3$ or $k = 1$.*

Remark 5.6. The only additional cases where the numerical value of $\text{genus}(\varepsilon_i)$ is known are:

- the already noted $\text{genus}(\varepsilon_2) = 1$, valid over any sphere, and coming from (16) and Corollary 5.4;

- Karasev-Landweber’s result that $\text{genus}(\varepsilon_3) = 1$ ([KL12]), valid over any k -dimensional sphere except perhaps if $k = 4 \cdot 3^e$ for some $e \geq 0$.

Proof of Proposition 5.5. The second assertion follows from the first one in view of Corollary 5.4. To prove the first assertion, we derive a contradiction from the assumption that ε_i admits a global continuous section σ . Consider the map $c: S^k \rightarrow (S^k)^{J_i}$ given as the composite

$$S^k \xrightarrow{\alpha} C_i(S^k) \xrightarrow{\tilde{\sigma}} e_i^{-1}(C_i(S^k)) \hookrightarrow (S^k)^{J_i}.$$

Here $\alpha(x) = (x, zx, z^2x, \dots, z^{i-1}x)$, where $z \in S^1$ is a primitive i^{th} root of unity acting on S^k in the standard way (recall k is odd), and $\tilde{\sigma}$ is the Σ_n -equivariant section of $e_i: e_i^{-1}(C_i(S^k)) \rightarrow C_i(S^k)$ obtained as the pull-back in (5) of the assumed σ . Thus, for each $x \in S^k$, $c(x)$ is a multipath $\{c_j(x)\}_{j=0}^{i-1} \in (S^k)^{J_i}$, where each $c_j(x)$ is a path in S^k starting at a point $s(x) \in S^k$ and ending at z^jx , for a continuous map $s: S^k \rightarrow S^k$. Note that the equivariance of $\tilde{\sigma}$ gives

$$(18) \quad c_j(zx) = c_{j+1}(x)$$

for all $x \in S^k$ —here the value of j is to be interpreted modulo i . Then the map $H: S^k \times [0, 1] \rightarrow S^k$ defined by $H(x, t) = c_0(x)(t)$ is a homotopy starting at s and ending at the identity. In particular, $s: S^k \rightarrow S^k$ has degree 1. The contradiction comes by observing that the degree of s would be divisible by i . Indeed, (18) gives

$$s(zx) = c_0(zx)(0) = c_1(x)(0) = s(x),$$

so that s factors as $S^k \xrightarrow{\text{proj}} L^k(i) \rightarrow S^k$ where $L^k(i)$ is the standard lens space $S^k/(\mathbb{Z}/i)$. \square

Corollary 5.7. *The known equality $\text{TC}_2^S(S^k) = 2$ (valid for any integer $k > 0$) extends as $\text{TC}_n^S(S^k) = 2(n-1)$ in the following two instances:*

- $n \geq 2$, provided $k = 1$.
- $n = 3$, provided k is odd.

6. CELLULAR STRATIFICATIONS

There have been several attempts at weakening the conditions for CW complexes in order to handle spaces with a more general type of decomposition into cells. Schürmann’s book [Sc03] is a good example of such an investigation. This section contains our proposal for such a goal—the main technical tool in preparation for the proof of Theorem 5.2.

In the first part of this section we introduce the notion of cellular stratified spaces, and describe some of their basic properties and connections with other more combinatorial structures. Our idea is to allow the use of cells that are not necessarily “closed” so that open manifolds such as complements of hyperplane arrangements and configuration spaces can be treated, up to homotopy, as finite CW complexes. Such a treatment is justified in the second part of this section, where we develop the tools leading to a proof of the key fact in the previous section (Theorem 5.2), namely that the homotopy properties of

a regular and polyhedrally normal cellular stratified space are captured by its barycentric subdivision.

6.1. Basic properties.

Definition 6.1.1. Let X be a topological space, and n a non-negative integer. An n -cell structure on a subspace $e \subset X$ consists of a pair (D, φ) formed by a subspace D of the closed n -disk D^n with $\text{Int}(D^n) \subset D$, and a continuous map $\varphi: D \rightarrow X$, satisfying the following conditions:

- (1) $\varphi(D) = \bar{e}$;
- (2) the restriction $\varphi|_{\text{Int}(D^n)}: \text{Int}(D^n) \rightarrow e$ is a homeomorphism;
- (3) the pair (D, φ) is maximal among pairs satisfying the above two conditions.

When the meaning is clear from the context, we refer to an n -cell structure (D, φ) on e just by e , in which case we also say that e is a cell of dimension n . The map φ is called the *characteristic map* of e , and D is called the *domain* for e .

When a subspace A of X contains a cell $e \subset X$ with structure (D, φ) , we will think of e also as a cell of A , with domain $D_A := \varphi^{-1}(\bar{e} \cap A)$ and characteristic map $\varphi|_{D_A}: D_A \rightarrow \bar{e} \cap A$ (of course $\bar{e} \cap A$ is the closure of e in A , and $\varphi|_{D_A}$ is surjective, as required in item (1) above).

Definition 6.1.2. Let X be a topological space. A *cellular stratification* \mathcal{C} on X is a filtration $X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots$ by subspaces of X satisfying the following conditions:

- (i) $X = \bigcup_{n \geq 0} X_n$;
- (ii) for $n \geq 0$, the subspace $X_n - X_{n-1}$ (where we put $X_{-1} = \emptyset$) decomposes as a topological disjoint union,

$$X_n - X_{n-1} = \coprod_{\lambda \in \Lambda_n} e_\lambda,$$

where each e_λ has an n -cell structure $(D_\lambda, \varphi_\lambda)$.

- (iii) (closure-finiteness) for each n -cell e_λ , $\partial e_\lambda := \bar{e}_\lambda - e_\lambda$ is covered by finitely many cells of dimension less than n ;
- (iv) (weak topology) X has the weak topology determined by the closures \bar{e}_λ for all $\lambda \in \Lambda_n$ and all $n \geq 0$.

We remark that when we say that \mathcal{C} is a cellular stratification on X , we mean a *rigged* such structure in the sense of [FR84, Section 1 of Chapter 2], i.e. we refer not only to the filtration $\{X_n\}$ but also to the fixed set of (domains and) characteristic maps of cells. We refer to X_n as the n^{th} *skeleton* of \mathcal{C} , and denote it by $\text{sk}_n(\mathcal{C})$, or by $\text{sk}_n(X)$ if the cellular stratification \mathcal{C} is clear from the context.

A *cellular stratified space* is a pair (X, \mathcal{C}) where \mathcal{C} is a cellular stratification of X . As usual, we abbreviate (X, \mathcal{C}) to X if there is no danger of confusion.

Definition 6.1.3. Let (X, \mathcal{C}) and (X', \mathcal{C}') be two cellular stratified spaces.

- (1) A union A of cells of \mathcal{C} is called a *cellular stratified subspace* of (X, \mathcal{C}) if $(A, \mathcal{C}|_A)$ becomes a cellular stratified space, where $\mathcal{C}|_A$ is the filtration $A_0 \subset A_1 \subset \cdots \subset A_n \subset \cdots$ with $A_n = A \cap X_n$ (and where domains and characteristic maps of cells in A are taken as indicated just before Definition 6.1.2).
- (2) A map $f : X \rightarrow X'$ is called an isomorphism of cellular stratified spaces (with respect to the stratifications \mathcal{C} and \mathcal{C}') if it is a homeomorphism so that for each n -cell e in \mathcal{C} there is an n -cell e' in \mathcal{C}' such that $D_e = D_{e'}$, $f(e) = e'$, and $\varphi_{e'} = f|_{\bar{e}} \circ \varphi_e$, where $\varphi : D_e \rightarrow \bar{e}$ and $\varphi' : D_{e'} \rightarrow \bar{e}'$ are the characteristic maps of e and e' respectively.
- (3) A map $f : X \rightarrow X'$ is called an embedding of cellular stratified spaces (with respect to \mathcal{C} and \mathcal{C}') if it is a topological embedding whose image $f(X)$ is a cellular stratified subspace of X' so that $f : X \rightarrow f(X)$ is an isomorphism with respect to the stratifications \mathcal{C} and $\mathcal{C}'|_{f(X)}$.

We usually impose further conditions on cellular stratified spaces.

Definition 6.1.4. Let $X = (X, \mathcal{C})$ be a cellular stratified space.

- a. We say that X is *finite* if the set of cells (of all dimensions) is finite. We say that X is of *finite type* if for every $n \geq 0$ the set of cells of dimension n is finite. (For instance, conditions (iii) and (iv) in Definition 6.1.2 hold for free in the case of a finite cellular stratified space. Consequently, any subspace A of X which is the union of finitely many cells in a cellular stratification \mathcal{C} of X is automatically a cellular stratified subspace with the filtration $\mathcal{C}|_A$.)
- b. A cell e_λ of X is said to be *regular* if its characteristic map is a homeomorphism onto \bar{e}_λ . Furthermore, X is called *regular* if all its cells are regular.
- c. An n -cell e_λ of X is said to be *closed* if $D_\lambda = D^n$, and X is usually called a *CW complex* if all its cells are closed.
- d. We call X *normal* if, for each n -cell e_λ , ∂e_λ is a union of finitely many cells of dimension less than n .
- e. A pair of cells (e_λ, e_μ) of X is said to be *strongly normal* provided $e_\mu \subset \bar{e}_\lambda$ and there exists an embedding $b_{\mu,\lambda} : D_\mu \rightarrow D_\lambda$ with $\varphi_\mu = \varphi_\lambda \circ b_{\mu,\lambda}$ (note that there is a unique such embedding $b_{\mu,\lambda}$ when e_λ is regular). We call X *strongly normal* if it is normal and all pairs of cells (e_λ, e_μ) with $e_\mu \subset \partial e_\lambda$ are strongly normal.
- f. We call X *totally normal* if it is normal and the following two conditions hold for each n and each n -cell e_λ :
 - there exists a structure of regular CW complex on S^{n-1} (which depends on the cell e_λ) containing $\partial D_\lambda := D_\lambda - \text{Int}(D^n)$ as a stratified subspace;
 - for any cell e in ∂D_λ there exists a cell e_μ contained in ∂e_λ having the same domain as e and such that the characteristic map $\varphi : D \rightarrow \bar{e}$ of e —a homeomorphism—lifts

φ_μ through φ_λ , that is, yields a commutative diagram

$$\begin{array}{ccc} \bar{e} & \xrightarrow{\varphi_\lambda|_{\bar{e}}} & \bar{e}_\mu \\ \varphi \uparrow & \nearrow \varphi_\mu & \\ D = D_\mu & & \end{array}$$

- g. Given a stratified subspace $A = (A, \mathcal{C}|_A)$ of X , we say that the pair (X, A) is *relatively regular* (respectively *relatively normal*, *relatively strongly normal*, *relatively totally normal*), if $X - A$ is a cellular stratified subspace of X which is regular (respectively normal, strongly normal, totally normal).

Each of the concepts in Definitions 6.1.1–6.1.4 can be illustrated by taking a difference $X - A$, for a suitably chosen subcomplex A of a CW complex X (Examples 7.2.1 and 7.2.2 in the next section describe particularly amenable situations). The reader will gain familiarity with Definitions 6.1.1–6.1.4 by filling in the easy details for the following assertions (for the third one keep in mind that a regular CW complex is normal, see [LW69, Section III.2]):

Example 6.1.5. A regular normal cellular stratified space is strongly normal. A totally normal cellular stratified space is strongly normal. A regular CW complex is totally normal.

We need two operations on cellular stratified spaces: subdivisions and coarsenings.

Definition 6.1.6. A *subdivision* of a cellular stratification \mathcal{C} on X is a cellular stratification \mathcal{C}' satisfying the following conditions for each cell e_λ in \mathcal{C} :

- (1) there exist a finite number of cells $e_{\mu_1}, \dots, e_{\mu_t}$ in \mathcal{C}' giving a disjoint union decomposition

$$e_\lambda = \bigcup_{i=1}^t e_{\mu_i};$$

- (2) there exists a regular cellular stratification on D_λ which contains $\text{Int}(D_\lambda)$ as a cellular stratified subspace, and such that cells in $\text{Int}(D_\lambda)$ are in one-to-one correspondence with $e_{\mu_1}, \dots, e_{\mu_t}$. Further, if $\varphi_\lambda : D_\lambda \rightarrow \bar{e}_\lambda$ is the characteristic map for e_λ and $\varphi'_i : D'_{\mu_i} \rightarrow \bar{e}'_{\mu_i}$ is the characteristic map for the cell e'_{μ_i} in $\text{Int}(D_\lambda)$ corresponding to e_{μ_i} ($1 \leq i \leq t$), then $D_{\mu_i} = D'_{\mu_i}$ and the following diagram commutes

$$\begin{array}{ccccc} D_\lambda & \xrightarrow{\varphi_\lambda} & \bar{e}_\lambda & \hookrightarrow & X \\ \uparrow & & \uparrow & & \\ D_{\mu_i} & \xrightarrow{\varphi_i} & \bar{e}_{\mu_i} & & \end{array}$$

where the inclusion on the left hand side is the composite of φ'_i (which is an embedding in view of the regularity of the stratification on D_λ) and the inclusion $\bar{e}'_{\mu_i} \hookrightarrow D_\lambda$.

In this case we also say that \mathcal{C} is a *coarsening* of \mathcal{C}' .

Although total normality is one of the most important properties a cellular stratified space can have, in practice it can turn out to be a difficult task to determine whether a given cellular stratification has this property. Our way around this problem is to focus on regular stratified subspaces of CW complexes which have a suitable ‘local’ polyhedral structure. With this aim, we start by recalling the following combinatorial structure, see [Ko08, Section 2.2.4] for details.

Definition 6.1.7. A *polyhedral complex* in \mathbb{R}^m is a subspace $K \subset \mathbb{R}^m$ equipped with a finite family of maps $\{\varphi_i : P_i \longrightarrow K\}_{i=1}^n$ satisfying the following conditions:

- (1) each P_i is a convex polytope in some euclidean space;
- (2) each φ_i is an affine equivalence onto its image;
- (3) $K = \bigcup_{i=1}^n \varphi_i(P_i)$;
- (4) for $i \neq j$, $\varphi_i(P_i) \cap \varphi_j(P_j)$ is a face of $\varphi_i(P_i)$ and of $\varphi_j(P_j)$.

The P_i ’s are called the *generating polytopes* of K .

Any polyhedral complex P has a canonical regular CW structure, that we call the associated polyhedral stratification on P . Note this is a totally normal cellular stratification. We generalize this fact (in Theorem 6.1.14 below) in two ways: (a) by weakening Definition 6.1.7 on a cell-by-cell basis (Definition 6.1.8 below), and (b) by drawing the total normality conclusion (in the slightly weaker form given in Definition 6.1.9) for any regular stratified subspace of the ambient complex.

Definition 6.1.8. Let X be a subspace of \mathbb{R}^m . A structure of normal CW complex on X is said to be *locally polyhedral* if, for each n -cell e with characteristic map φ , there exists a polyhedral complex P and a homeomorphism $\alpha : P \rightarrow D^n$ such that the composite

$$P \xrightarrow{\alpha} D^n \xrightarrow{\varphi} X \subset \mathbb{R}^m$$

is a PL map. The map α is called a *polyhedral replacement* for φ . (Note that any polyhedral complex is locally polyhedral.)

Definition 6.1.9. A normal cellular stratified space (X, \mathcal{C}) is said to be *polyhedrally normal* if, for each cell $\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda}$, there exists a polyhedral complex P_λ and a homeomorphism $\alpha_\lambda : P_\lambda \rightarrow D^{\dim e_\lambda}$ (called a *polyhedral replacement* of φ_λ) satisfying the following conditions:

- (1) There is a coarsening \mathcal{C}_λ of the associated polyhedral stratification on P_λ containing $\alpha_\lambda^{-1}(\partial D_\lambda)$ as a stratified subspace and $\alpha_\lambda^{-1}(\text{Int}(D_\lambda))$ as a single cell.

- (2) For any cell e in $\mathcal{C}_\lambda|_{\alpha_\lambda^{-1}(D_\lambda)}$, there is a cell e_μ in \mathcal{C} contained in $\overline{e_\lambda}$ and a PL map $b_e : \alpha_\mu^{-1}(D_\mu) \rightarrow \alpha_\lambda^{-1}(D_\lambda)$ such that the diagram

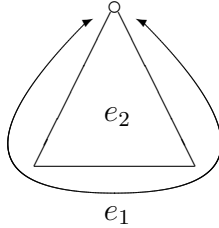
$$\begin{array}{ccccc}
 \alpha_\lambda^{-1}(D_\lambda) & \xrightarrow{\alpha_\lambda} & D_\lambda & \xrightarrow{\varphi_\lambda} & \overline{e_\lambda} \\
 \uparrow b_e & & & & \uparrow \\
 \alpha_\mu^{-1}(D_\mu) & \xrightarrow{\alpha_\mu} & D_\mu & \xrightarrow{\varphi_\mu} & \overline{e_\mu}
 \end{array}$$

is commutative, and that the composition

$$b_e \circ (\alpha_\mu^{-1}|_{D_\mu}) : D_\mu \rightarrow \alpha_\lambda^{-1}(D_\lambda)$$

is the characteristic map of e . (Note that if $e = \alpha_\lambda^{-1}(\text{Int}(D_\lambda))$, we must have $\mu = \lambda$ with b_e the identity map.)

Example 6.1.10. Consider the disk D^2 with the stratification \mathcal{C}_0 consisting of the three cells $e^0 = \{(0, 1)\}$, $e^1 = S^1 - \{(0, 1)\}$, and $e^2 = \text{Int}(D^2)$. Note that (D^2, \mathcal{C}_0) is a non-regular locally polyhedral CW complex. Indeed, $\alpha_2 : \Delta^2 \rightarrow D^2$ can be taken to be any homeomorphism so that $\alpha_2^{-1}(0, 1)$ is a vertex of Δ^2 , while the domain of the characteristic map for e^1 gets subdivided into three 1-cells and four 0-cells. On the other hand, let (T, \mathcal{C}) be the stratified subspace consisting of e^1 and e^2 (depicted in the diagram below). The two polyhedral replacements above show that (T, \mathcal{C}) is a regular and polyhedrally normal cellular stratified space which, however, fails to be totally normal due to the lack of regularity of \mathcal{C}_0 . For future reference, it will also be convenient to consider the totally normal stratification \mathcal{C}' on T induced, via α_2 , after removing $\alpha_2^{-1}(0, 1)$ from the 2-simplex Δ^2 (thus \mathcal{C}' consists of two 0-cells, three 1-cells, and a 2-cell).



Remark 6.1.11. Note that we do not require the stratification \mathcal{C}_λ in Definition 6.1.9 to be regular—not even on $\partial D^{\dim(e_\lambda)}$. Apart from the PL requirements on the maps b_e , this is the main difference between Definition 6.1.9 and the concept of total normality in item f. of Definition 6.1.4. Indeed, our concept of polyhedral normality is a suitable combination of *local polyhedrality* and *total normality*.

Remark 6.1.12. Polyhedrally normal cellular stratified spaces generalize PLCW complexes defined and studied in [Kir11] by Alexander Kirillov, Jr.

Lemma 6.1.13. *Assume that the cellular stratified space (X, \mathcal{C}) in Definition 6.1.9 is in addition regular. Then the maps b_e in item (2) of that definition are embeddings, and each homeomorphism $\varphi_\lambda \circ \alpha_\lambda$ is an isomorphism of cellular stratified spaces with respect to $\mathcal{C}_\lambda|_{\alpha_\lambda^{-1}(D_\lambda)}$ and $\mathcal{C}|_{\bar{e}_\lambda}$. Furthermore:*

- (1) *The cellular stratified space X is strongly normal.*
- (2) *Each $\mathcal{C}_\lambda|_{\alpha_\lambda^{-1}(D_\lambda)}$ is normal.*
- (3) *The maps b_e are embeddings of cellular stratified spaces with respect to the coarser structures $\mathcal{C}_\mu|_{\alpha_\mu^{-1}(D_\mu)}$ and $\mathcal{C}_\lambda|_{\alpha_\lambda^{-1}(D_\lambda)}$.*

Proof. The first two assertions follow directly from the diagram in item (2) of Definition 6.1.9. For the strong normality of X , it suffices to take $b_{\mu,\lambda} = \alpha_\lambda \circ b_e \circ (\alpha_\mu)^{-1}$ in item e. of Definition 6.1.4. The normality of $\mathcal{C}_\lambda|_{\alpha_\lambda^{-1}(D_\lambda)}$ is a consequence of that for $\mathcal{C}|_{\bar{e}_\lambda}$. Finally, the maps b_e are embeddings of cellular stratified spaces as indicated because each inclusion $\bar{e}_\mu \hookrightarrow \bar{e}_\lambda$ identifies \bar{e}_μ as a cellular stratified subspace of \bar{e}_λ . \square

Theorem 6.1.14. *Any regular stratified subspace of a locally polyhedral CW complex is polyhedrally normal.*

We use the following well-known facts in PL topology in order to prove Theorem 6.1.14.

Lemma 6.1.15. *Given a convex polytope P , any PL map $f: \text{Int}(P) \rightarrow \mathbb{R}^n$ admits a unique PL extension $\tilde{f}: P \rightarrow \mathbb{R}^n$.*

Proof. This is obvious, since PL maps are locally affine and affine maps can be uniquely extended to the boundaries. \square

Lemma 6.1.16. *Let K and L be polyhedral complexes. For any PL map $f: K \rightarrow L$, there exist simplicial subdivisions K' and L' of K and L , respectively, such that the induced map $f: K' \rightarrow L'$ is simplicial.*

Proof. Apply [RS72, Theorem 2.14] noticing that any polyhedral complex has a simplicial subdivision. \square

Proof of Theorem 6.1.14. Let A be a regular stratified subspace of a locally polyhedral CW complex X . Since the normality of A is a consequence of that of X , the proof will be complete as soon as we find a polyhedral replacement for each cell in A satisfying the conditions in Definition 6.1.9.

For an n -cell e in A , let $\varphi: D^n \rightarrow X$ be the characteristic map of e regarded as a cell of X , and recall that the characteristic map of e as a cell of A is given by

$$\varphi_A = \varphi|_{D_A}: D_A = \varphi^{-1}(\bar{e} \cap A) \longrightarrow A.$$

By assumption, there exists a polyhedral complex P and a homeomorphism $\alpha: P \rightarrow D^n$. By normality, we have

$$(19) \quad \bar{e} \cap A = e \cup \bigcup_{i=1}^{\ell} e_i$$

for a finite number of cells e_i of lower dimensions. For each such cell e_i , say of dimension k , consider the commutative diagram

$$\begin{array}{ccccc}
 P & \xrightarrow{\alpha} & D^n & \xrightarrow{\varphi} & \bar{e} \\
 & & \uparrow & & \uparrow \\
 & & D_A & \xrightarrow{\varphi_A} & \bar{e} \cap A \\
 b_i \uparrow & & \uparrow & & \uparrow \\
 P'_i & \xrightarrow{\alpha_i|_{P'_i}} & D_i & \xrightarrow{\varphi_i} & \bar{e}_i \cap A
 \end{array}$$

where $\varphi_i : D_i \rightarrow \bar{e}_i \cap A$ is the characteristic map of e_i in A , $\alpha_i : P_i \rightarrow D^k$ is a polyhedral replacement for the characteristic map of e_i (in X), and P'_i is the (dense) subspace of P_i defined by $P'_i = \alpha_i^{-1}(D_i)$. By the regularity of A , φ_A is a homeomorphism and we obtain the dotted lift $D_i \rightarrow D_A$ of φ_i . We also obtain a unique map $b_i : P'_i \rightarrow P$ making the diagram commutative, since α is a homeomorphism. By assumption, the composites $\varphi_i \circ \alpha_i$ and $\varphi \circ \alpha$ are PL, and hence b_i is also a PL map. By Lemma 6.1.15, we obtain a PL map

$$\beta_i : P_i = \overline{P'_i} \longrightarrow P$$

extending b_i . By Lemma 6.1.16, there exist simplicial subdivisions P'' and P''_i of P and P_i , respectively, in terms of which β_i is a simplicial map for all i (in particular each $\beta_i(P''_i)$ is a subcomplex of P''). Define $\alpha'' : P'' \rightarrow D^n$ to be the composition of the identification $P'' \cong P$ and α . We now show that α'' works as a polyhedral replacement of φ . By (19) and the regularity of e (in A), we get a stratification

$$(\alpha'')^{-1}(D_A) = \alpha^{-1}(\text{Int}(D^n)) \cup \bigcup_{i=1}^{\ell} b_i(\alpha_i^{-1}(\text{Int}(D_i))),$$

which is made cellular by defining $b_i \circ (\alpha_i|_{P'_i})^{-1}$ and α^{-1} to be the characteristic maps for $b_i(\alpha_i^{-1}(\text{Int}(D_i)))$ and $\alpha^{-1}(\text{Int}(D^n))$ respectively. Since $\beta_i(P''_i)$ is a subcomplex of P'' , this procedure can actually be extended to get a coarsening of P'' satisfying the conditions in Definition 6.1.9. Indeed, just add all cells not in the image of any b_i . \square

6.2. Barycentric subdivision. It is a standard technique in combinatorial algebraic topology to translate geometric properties of a regular CW complex into combinatorial properties of its face poset. The goal of Definitions 6.1.1–6.1.4, 6.1.6–6.1.8 and 6.1.9 is to isolate the critical features that allow us to extend the above fruitful interaction to the case of cellular stratified spaces.

Definition 6.2.1. Let (X, \mathcal{C}) be a regular cellular stratified space.

1. The *face poset* of (X, \mathcal{C}) is the set $F(X, \mathcal{C}) = \{e \mid e \text{ is a cell in } \mathcal{C}\}$ with partial order defined by $e' \leq e$ whenever $e' \subset \bar{e}$. As usual, we write $e' < e$ to mean $e' \leq e$ with $e' \neq e$.

2. The *barycentric subdivision* of (X, \mathcal{C}) , denoted by $\text{Sd}(X, \mathcal{C})$, is the geometric realization of $\Delta F(X, \mathcal{C})$ —the order complex of $F(X, \mathcal{C})$.

Remark 6.2.2. Given a poset P , the order complex ΔP of P is defined to be the ordered simplicial complex consisting of finite totally ordered subsets of P . An element $\mathbf{p} \in \Delta P$ having set-cardinality $k+1$ is called a k -chain in P , and its elements will be specified in the form $\mathbf{p}: p_0 < p_1 < \cdots < p_k$. When P is regarded as a small category, it is well known that the geometric realization of ΔP (denoted by $|\Delta P|$) coincides with the classifying space BP of P . Thus

$$\text{Sd}(X, \mathcal{C}) = BF(X, \mathcal{C}) = |\Delta F(X, \mathcal{C})|.$$

When X or \mathcal{C} is obvious from the context, we use the shorthand $F(\mathcal{C})$ or $F(X)$ instead of $F(X, \mathcal{C})$, and $\text{Sd}(\mathcal{C})$ or $\text{Sd}(X)$ instead of $\text{Sd}(X, \mathcal{C})$.

Remark 6.2.3. Definition 6.2.1 does not quite describe the “right” geometric object (in the sense of Theorem 6.2.4 below) when \mathcal{C} is non-regular. The current statements are enough for the purposes of this paper.

It is well known (see for instance [BLSWZ99, Proposition 4.7.8]) that, if \mathcal{C} is a regular CW complex structure on a space X , then $\text{Sd}(\mathcal{C})$ is homeomorphic to X . As a CW complex, $\text{Sd}(\mathcal{C})$ gives the usual barycentric subdivision of \mathcal{C} . However $\text{Sd}(\mathcal{C})$ is usually much thinner than X when \mathcal{C} is just a regular cellular stratification on X . For instance, if X is an open disk with exactly one n -cell and characteristic map $\varphi = \text{id}: \text{Int}(D^n) \rightarrow \text{Int}(D^n)$, then $\text{Sd}(\mathcal{C})$ is a single point—which is homotopy equivalent but not homeomorphic to X . In fact, the next result, whose proof is the central goal of this section, asserts that, under suitable conditions, no homotopy property is lost when X is replaced by the combinatorial model $\text{Sd}(\mathcal{C})$.

Theorem 6.2.4. *For a regular totally normal cellular stratification \mathcal{C} on X , the barycentric subdivision $\text{Sd}(\mathcal{C})$ can be embedded in X as a strong deformation retract. When \mathcal{C} is a regular CW complex structure, the embedding is a homeomorphism. Furthermore, when X is equipped with a cellular action of a group G , both the embedding and the deformation retraction can be taken to be G -equivariant. Indeed, these two constructions can be made to be natural with respect to embeddings of cellular stratified spaces.*

In preparation for the proof of Theorem 6.2.4, we make explicit the construction of the stated embedding (which is a straightforward extension of the situation for regular CW complexes).

Lemma 6.2.5. *Let \mathcal{C} be a strongly normal cellular stratification on X . Then for each non-negative integer k and each k -chain*

$$(20) \quad \mathbf{e}: e_{\lambda_0} < e_{\lambda_1} < \cdots < e_{\lambda_k}$$

in $F(\mathcal{C})$ —that is, for each k -simplex $\Delta^k = \Delta^k(\mathbf{e})$ in $|\Delta F(\mathcal{C})|$ —there exist embeddings

$$(21) \quad d_{\mathbf{e}}: \Delta^k \rightarrow D_{\lambda_k} \quad \text{and} \quad i_{\mathbf{e}}: \Delta^k \rightarrow \overline{e_{\lambda_k}} \subset X$$

satisfying the following conditions:

- (i) $i_{\mathbf{e}} = \varphi_{\lambda_k} \circ d_{\mathbf{e}}$, where $\varphi_{\lambda_k} : D_{\lambda_k} \rightarrow \overline{e_{\lambda_k}} \subset X$ stands for the characteristic map of e_{λ_k} .
- (ii) For $0 \leq j \leq k$ let \mathbf{e}_j denote the $(k-1)$ -subchain $e_{\lambda_0} < \cdots < \widehat{e_{\lambda_j}} < \cdots < e_{\lambda_k}$ of (20), where the cell e_{λ_j} has been removed. Then the restriction of $i_{\mathbf{e}}$ to the face of Δ^k opposite to the vertex of $\Delta F(\mathcal{C})$ corresponding to e_{λ_j} coincides with $i_{\mathbf{e}_j}$.

Corollary 6.2.6. *The embeddings $\{i_{\mathbf{e}}\}$ in Lemma 6.2.5 fit together assembling an embedding $i : \text{Sd}(\mathcal{C}) \hookrightarrow X$ which is natural for embeddings of cellular stratified spaces.*

Proof of Lemma 6.2.5 and Corollary 6.2.6. We construct the required embeddings $d_{\mathbf{e}}$ and $i_{\mathbf{e}}$ by induction on k . The vertices of $\text{Sd}(\mathcal{C})$ are in one-to-one correspondence with cells in \mathcal{C} . For each cell e_{λ} , set $v_{\lambda} = \varphi_{\lambda}(0)$. This defines an obvious embedding $i_0 : \text{sk}_0 \text{Sd}(\mathcal{C}) \rightarrow X$, where $\text{sk}_{\ell} \text{Sd}(\mathcal{C})$ stands for the ℓ^{th} skeleton of $\text{Sd}(\mathcal{C})$.

Suppose we have constructed embeddings as required for each j -chain with $j < k$. For a k -chain (20), we next define embeddings (21) satisfying the above properties. By hypothesis (Definition 6.1.4.e) there is an embedding $b_{\lambda_{k-1}, \lambda_k} : D_{\lambda_{k-1}} \rightarrow \partial D_{\lambda_k}$ with $\varphi_{\lambda_{k-1}} = \varphi_{\lambda_k} \circ b_{\lambda_{k-1}, \lambda_k}$. By the inductive assumption, we have an embedding $i_{\mathbf{e}'} : \Delta^{k-1} \rightarrow \overline{e_{\lambda_{k-1}}} \subset X$ corresponding to the $(k-1)$ -chain $\mathbf{e}' : e_{\lambda_0} < \cdots < e_{\lambda_{k-1}}$, and an embedding $d_{\mathbf{e}'} : \Delta^{k-1} \rightarrow D_{\lambda_{k-1}}$ with $i_{\mathbf{e}'} = \varphi_{\lambda_{k-1}} \circ d_{\mathbf{e}'}$. Extending the composite $b_{\lambda_{k-1}, \lambda_k} \circ d_{\mathbf{e}'}$ to the joins yields the first map in the composite of embeddings

$$\Delta^k = \Delta^{k-1} * e_{\lambda_k} \rightarrow b_{\lambda_{k-1}, \lambda_k}(d_{\mathbf{e}'}(\Delta^{k-1})) * 0 \rightarrow D_{\lambda_k}.$$

This works as the first embedding in (21), while the second embedding is forced from (i). Condition (ii) as well as Corollary 6.2.6 are obvious from the construction. \square

At the end of the section we deduce Theorem 6.2.4 from a key special case (Lemma 6.2.10), where attention is focused on the case of a disk. In turn, our proof of the special case uses standard techniques in simplicial topology (Lemmas 6.2.8 and 6.2.9 below) based on the following concept.

Definition 6.2.7. Let K be a cellular stratified space. For $x \in K$, the *open star* around x in K , $\text{St}(x; K)$, is the union of those cells whose closure contains x . For a subset $A \subset K$, define

$$\text{St}(A; K) = \bigcup_{x \in A} \text{St}(x; K).$$

When K is a simplicial complex and A is a subcomplex, $\text{St}(A; K)$ is called the *regular neighborhood* of A in K .

Lemma 6.2.8. *Let K be a regular CW complex. For any stratified subspace L of K , the image of the regular neighborhood $\text{St}(\text{Sd}(L); \text{Sd}(\overline{L}))$ of $\text{Sd}(L)$ in $\text{Sd}(\overline{L})$ under the embedding (actually a homeomorphism) $i : \text{Sd}(K) \hookrightarrow K$ in Corollary 6.2.6 contains L .*

Proof. For a point $x \in L$, there exists a cell e in L with $x \in e$. Under the barycentric subdivision of \overline{L} , e is triangulated, namely there exists an n -chain $\mathbf{e} : e_0 < e_1 < \cdots < e_n =$

e of cells in \bar{L} such that $x \in i_e(\text{Int}(\Delta^n))$ and $v(e) \in \overline{i_e(\text{Int}(\Delta^n))}$ where $v(e)$ is the vertex in $\text{Sd}(L)$ corresponding to e . By definition of St , we have

$$i_e(\text{Int}(\Delta^n)) \subset i(\text{St}(v(e); \text{Sd}(\bar{L}))) \subset i(\text{St}(\text{Sd}(L); \text{Sd}(\bar{L}))),$$

so that $L \subset i(\text{St}(\text{Sd}(L); \text{Sd}(\bar{L})))$. \square

It follows from the construction of the barycentric subdivision that, under the conditions of the lemma, $\text{Sd}(L)$ is a full subcomplex of $\text{Sd}(\bar{L})$, that is, for any collection of vertices v_0, \dots, v_k in $\text{Sd}(L)$ which forms a simplex σ in $\text{Sd}(\bar{L})$, the simplex σ belongs to $\text{Sd}(L)$. The important property of such a situation is, in general, that a full subcomplex A of a simplicial complex K is a simplicial strong deformation retract of its regular neighborhood $\text{St}(A; K)$. Indeed, as shown in [ES52, Lemma II.9.3] (see alternatively the case $K' = \emptyset$ in the proof of Lemma 6.2.9 below), there is a simplicial homotopy rel A between the identity on $\text{St}(A; K)$ and the composite

$$(22) \quad \text{St}(A; K) \xrightarrow{r_A} A \xrightarrow{\iota_A} \text{St}(A; K).$$

Here r_A is the retraction given by

$$r_A(x) = \frac{1}{\sum_{v \in A \cap \sigma} t(v)} \sum_{v \in A \cap \sigma} t(v)v$$

whenever $x = \sum_{v \in \sigma} t(v)v$ belongs to a simplex σ . Further, by a simplicial homotopy H on $\text{St}(A; K)$ we mean one for which, whenever a point x lies in a cell e of $\text{St}(A; K)$, the curve $H(x, s)$ stays in e for $s < 1$. The above basic property can be extended as follows:

Lemma 6.2.9. *Let A and K' be subcomplexes of a finite simplicial complex K , with A a full subcomplex. Set $A' := A \cap K'$. Then any simplicial homotopy H' rel A' between the identity on $\text{St}(A'; K')$ and $\iota_{A'} \circ r_{A'}$ can be extended to a simplicial homotopy H rel A between the identity on $\text{St}(A; K)$ and the map in (22). Furthermore, assume that M is a cellular stratified subspace of $\text{St}(A; K)$ containing A , and that M' is a cellular stratified subspace of $\text{St}(A'; K') \cap M$ containing A' . Then any simplicial homotopy rel A' between the identity on M' and the composite*

$$M' \hookrightarrow \text{St}(A'; K') \xrightarrow{r_{A'}} A' \hookrightarrow M'$$

can be extended to a simplicial homotopy rel A between the identity on M and the composite

$$M \hookrightarrow \text{St}(A; K) \xrightarrow{r_A} A \hookrightarrow M.$$

Proof. We regard K as a subcomplex of a large simplex S , and let $V(B)$ denote the set of vertices of a subcomplex B of S . Then every point $x \in |K|$ can be expressed as a formal convex combination

$$x = \sum_{v \in V(K)} a_v v$$

with $\sum_{v \in V(K)} a_v = 1$ and $a_v \geq 0$.

Let $H' : \text{St}(A'; K') \times [0, 1] \rightarrow \text{St}(A'; K')$ be a simplicial homotopy rel A' between the identity on $\text{St}(A'; K')$ and $\iota_{A'} \circ r_{A'}$. Consider the homotopy

$$(23) \quad H : \text{St}(A; K) \times [0, 1] \rightarrow \text{St}(A; K)$$

defined by

$$\begin{aligned} H(x, s) = & \frac{\alpha + (1-s)\beta}{(1-s) + s(\alpha + \gamma)} H' \left(\sum_i \frac{a_i}{\alpha + \beta} u'_i + \sum_j \frac{b_j}{\alpha + \beta} v'_j, s \right) \\ & + \sum_k \frac{c_k}{(1-s) + s(\alpha + \gamma)} u_k + \sum_\ell \frac{(1-s)d_\ell}{(1-s) + s(\alpha + \gamma)} v_\ell, \end{aligned}$$

where $s \in [0, 1]$, $\alpha = \sum_i a_i$, $\beta = \sum_j b_j$, $\gamma = \sum_k c_k$, and $x \in \text{St}(A; K)$ has the form $x = \sum_i a_i u'_i + \sum_j b_j v'_j + \sum_k c_k u_k + \sum_\ell d_\ell v_\ell$ with

- $u'_i \in V(A')$,
- $v'_j \in V(K') - V(A')$,
- $u_k \in V(A) - V(A')$,
- $v_\ell \in V(K) - (V(K') \cup V(A))$.

Then

$$\begin{aligned} H(x, 0) &= (\alpha + \beta) \left(\sum_i \frac{a_i}{\alpha + \beta} u'_i + \sum_j \frac{b_j}{\alpha + \beta} v'_j \right) + \sum_k c_k u_k + \sum_\ell d_\ell v_\ell \\ &= \sum_i a_i u'_i + \sum_j b_j v'_j + \sum_k c_k u_k + \sum_\ell d_\ell v_\ell \\ &= x, \\ H(x, 1) &= \frac{\alpha}{\alpha + \gamma} r_{A'} \left(\sum_i \frac{a_i}{\alpha + \beta} u'_i + \sum_j \frac{b_j}{\alpha + \beta} v'_j \right) + \sum_k \frac{c_k}{\alpha + \gamma} u_k \\ &= \frac{\alpha}{\alpha + \gamma} \sum_i \left(\frac{\frac{a_i}{\alpha + \beta}}{\frac{\alpha}{\alpha + \beta}} \right) u'_i + \sum_k \frac{c_k}{\alpha + \gamma} u_k \\ &= \sum_i \frac{a_i}{\alpha + \gamma} u'_i + \sum_k \frac{c_k}{\alpha + \gamma} u_k \\ &= r_A(x). \end{aligned}$$

Further, when $x \in K'$, we have $c_k = d_\ell = 0$ and $x = \sum_i a_i u'_i + \sum_j b_j v'_j$. Since $\alpha + \beta = 1$, we then have

$$\begin{aligned} H(x, s) &= \frac{\alpha + (1-s)\beta}{(1-s) + s\alpha} H' \left(\sum_i a_i u'_i + \sum_j b_j v'_j, s \right) \\ &= \frac{1-s\beta}{1-s(1-\alpha)} H'(x, s) = H'(x, s). \end{aligned}$$

Lastly, when $x \in A$, we have $b_j = d_\ell = 0$ and $x = \sum_i a_i u'_i + \sum_k c_k u_k$. Since $\alpha + \gamma = 1$, we then have

$$\begin{aligned} H(x, s) &= \alpha H' \left(\sum_i \frac{a_i}{\alpha} u'_i, s \right) + \sum_k c_k u_k \\ &= \sum_i a_i u'_i + \sum_k c_k u_k = x. \end{aligned}$$

Thus H is the homotopy required in the first assertion of the lemma. The last assertion in the lemma follows since the homotopy (23) is defined on a cell-by-cell basis. \square

We are now ready to deduce a key special case of Theorem 6.2.4. Suppose \mathcal{C} is a regular CW complex structure on S^{n-1} , and $L \subset S^{n-1}$ is a stratified subspace. Set $K = L \cup \text{Int}(D^n)$ with the obvious stratification obtained from that in L by adding $\text{Int}(D^n)$ as an n -cell (this is a stratified subspace of the regular CW complex decomposition on D^n coming from \mathcal{C} by adding $\text{Int}(D^n)$ as an n -cell).

Corollary 6.2.10. *In the above situation, any simplicial homotopy relative to $i(\text{Sd}(L))$ between the identity on L and the composite*

$$L \hookrightarrow i(\text{St}(\text{Sd}(L); \text{Sd}(\overline{L}))) \xrightarrow{r_{i(\text{Sd}(L))}} i(\text{Sd}(L)) \hookrightarrow L$$

can be extended to a simplicial homotopy relative to $i(\text{Sd}(K))$ between the identity on K and the composite

$$K \hookrightarrow i(\text{St}(\text{Sd}(K); \text{Sd}(\overline{K}))) \xrightarrow{r_{i(\text{Sd}(K))}} i(\text{Sd}(K)) \hookrightarrow K.$$

Proof. We have observed that $\text{Sd}(K)$ is a full subcomplex of $\text{Sd}(\overline{K})$. Moreover, in view of Lemma 6.2.8, K can be considered as a cellular stratified subspace of the regular neighborhood of $\text{Sd}(K)$ in $\text{Sd}(\overline{K})$. Likewise, L can be considered as a cellular stratified subspace of the regular neighborhood $\text{St}(\text{Sd}(L); \text{Sd}(\overline{L})) \cap K$. Therefore Lemma 6.2.9 can be applied and the result follows. \square

Proof of Theorem 6.2.4. For each k -cell e_λ of X , let $\mathcal{C}_{\lambda,k}$ be a fixed regular CW complex structure on S^{k-1} as in the definition of total normality. We construct, by induction on k , strong deformation retractions

$$(24) \quad H_k: \text{sk}_k X \times [0, 1] \rightarrow \text{sk}_k X$$

of $\text{sk}_k X$ onto $i(\text{Sd}(\text{sk}_k \mathcal{C}))$ with the following property: For each k -cell e_λ , the restriction $H_k|_{\overline{e_\lambda} \times [0, 1]}$ lands in $\overline{e_\lambda}$ giving, in terms of the corresponding characteristic map $\varphi_\lambda: D_\lambda \rightarrow \overline{e_\lambda}$ (which is a homeomorphism), a simplicial homotopy $\text{rel } i(\text{Sd}(D_\lambda))$ between the identity on D_λ and the composite

$$(25) \quad D_\lambda \hookrightarrow i(\text{St}(\text{Sd}(D_\lambda); \text{Sd}(\overline{D_\lambda}))) \xrightarrow{r_{i(\text{Sd}(D_\lambda))}} i(\text{Sd}(D_\lambda)) \hookrightarrow D_\lambda$$

—here $\text{Sd}(D_\lambda)$ is taken with respect to the obvious stratified structure coming from $\mathcal{C}_{\lambda,k}|_{\partial D_\lambda}$ by adding the k -cell $\text{Int}(D^k)$.

When $k = 0$, there is nothing to prove, since $\text{Sd}(\text{sk}_0(X)) = \text{sk}_0 X$. Assuming we have constructed the required H_{k-1} , we next extend it to all k -cells. Using Corollary 6.2.10, we obtain, for each k -cell e_λ , a simplicial homotopy $H_\lambda : D_\lambda \times [0, 1] \rightarrow D_\lambda$ satisfying the two conditions:

- a. H_λ is a homotopy rel $i(\text{Sd}(D_\lambda))$ between the identity on D_λ and (25).
- b. H_λ extends the homotopy $H_{k-1,\lambda}$ given by the composite

$$\partial D_\lambda \times [0, 1] \xrightarrow{(\varphi_\lambda|_{\partial D_\lambda}) \times [0, 1]} \partial e_\lambda \times [0, 1] \xrightarrow{H_{k-1}|_{\partial e_\lambda \times [0, 1]}} \partial e_\lambda \xrightarrow{(\varphi_\lambda|_{\partial D_\lambda})^{-1}} \partial D_\lambda$$

(the fact that the middle map lands in ∂e_λ follows from the inductive construction).

By the regularity hypothesis, H_{k-1} and the various H_λ fit together to produce the new required homotopy (24), completing the inductive step in the construction of the strong deformation retraction of X onto $\text{Sd}(X)$.

Naturality of the homotopies with respect to embeddings of cellular stratified spaces follows since our construction is done on a cell-by-cell basis. In particular, if X is equipped with a cellular action of a group G , we obtain G -equivariant homotopies. \square

The regular polyhedrally normal cellular stratified space (T, \mathcal{C}) in Example 6.1.10 is not totally normal, so Theorem 6.2.4 does not apply directly. Yet, the deformation retraction asserted in that result still holds. The situation is illustrated in the following two pictures (where $\text{Sd}(\mathcal{C})$ is represented by the dotted line joining the barycenters of e_1 and e_2) as an instance of the general argument proving Theorem 6.2.11 below.



The idea is to construct the required strong deformation retraction in two stages. In the case of Example 6.1.10, the first step (illustrated in the left hand side picture above) is to collapse T onto the barycentric subdivision $\text{Sd}(\mathcal{C}')$. The latter can then, as a second step (illustrated in the right hand side picture above), be deformed onto $\text{Sd}(\mathcal{C})$ by linearly extending the deformation of the 1-cell onto its barycenter.

Theorem 6.2.11. *The first and third assertions in Theorem 6.2.4 are true for a finite regular polyhedrally normal cellular stratified space (X, \mathcal{C}) and a finite group G .*

As in Theorem 6.2.4, Theorem 6.2.11 refers to the embedding $i : \text{Sd}(\mathcal{C}) \hookrightarrow X$ in Corollary 6.2.6, which exists in view of item (1) in Lemma 6.1.13. Therefore it remains to construct, under the new hypothesis and on a cell-by-cell basis, the required strong deformation retraction.

Here is a summary of hypotheses, facts, and notation in preparation for the proof of Theorem 6.2.11: Since (X, \mathcal{C}) is assumed to be polyhedrally normal, each of its cells

$\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda}$ has a polyhedral replacement (a homeomorphism) $\alpha_\lambda : P_\lambda \rightarrow D^{\dim e_\lambda}$. For simplicity, we identify $\alpha_\lambda^{-1}(D_\lambda)$ with D_λ . As indicated in Lemma 6.1.13, for each pair of cells $e_\mu \subset \overline{e_\lambda}$, the PL map $b_{\mu,\lambda} : D_\mu \rightarrow D_\lambda$ in Definition 6.1.9 (satisfying $\varphi_\lambda \circ b_{\mu,\lambda} = \varphi_\mu$) is in fact an embedding. For convenience in the proof of Theorem 6.2.11, we are using here the more explicit notation “ $b_{\mu,\lambda}$ ” for what is denoted simply by “ b_e ” in Definition 6.1.9. Here e is the cell corresponding to e_μ under the isomorphism of cellular stratified spaces $\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda}$ (cf. Lemma 6.1.13 recalling that we are identifying $\alpha_\lambda^{-1}(D_\lambda)$ with D_λ). We also have, on each D_λ , a cellular stratification \mathcal{C}_λ which is a coarsening of the cellular stratification \mathcal{C}'_λ given by the restriction of the polyhedral cell decomposition of P_λ to D_λ . Note that \mathcal{C}_λ is normal and $b_{\mu,\lambda} : (D_\mu, \mathcal{C}_\mu) \rightarrow (D_\lambda, \mathcal{C}_\lambda)$ is an embedding of cellular stratified spaces in view of Lemma 6.1.13. On the other hand, there is no guarantee that each $b_{\mu,\lambda}$ is an embedding of cellular stratified spaces with respect to the corresponding finer cellular stratifications \mathcal{C}'_μ and \mathcal{C}'_λ . By the finiteness assumption, however, we may take common subdivisions and assume that each map

$$(26) \quad b_{\mu,\lambda} : (D_\mu, \mathcal{C}'_\mu) \rightarrow (D_\lambda, \mathcal{C}'_\lambda)$$

is such an embedding. This yields a regular and totally normal cellular stratification \mathcal{C}' on X which is a subdivision of \mathcal{C} .

Proof of Theorem 6.2.11. We first apply Theorem 6.2.4 to (X, \mathcal{C}') and obtain a strong deformation retraction of X onto $i(\text{Sd}(X, \mathcal{C}'))$

$$(27) \quad G : X \times [0, 1] \longrightarrow X.$$

Thus the main goal in the proof is to construct, for each cell e_λ in \mathcal{C} , a PL homotopy

$$(28) \quad H_\lambda : i(\text{Sd}(D_\lambda, \mathcal{C}'_\lambda)) \times [0, 1] \longrightarrow i(\text{Sd}(D_\lambda, \mathcal{C}_\lambda))$$

satisfying the following conditions:

- $i(\text{Sd}(D_\lambda, \mathcal{C}'_\lambda))$ strongly deformation retracts onto $i(\text{Sd}(D_\lambda, \mathcal{C}_\lambda))$ under H_λ .
- The following diagram is commutative for each cell e_μ in ∂e_λ :

$$(29) \quad \begin{array}{ccc} i(\text{Sd}(D_\lambda, \mathcal{C}'_\lambda)) \times [0, 1] & \xrightarrow{H_\lambda} & i(\text{Sd}(D_\lambda, \mathcal{C}_\lambda)) \\ \uparrow b_{\mu,\lambda} \times 1 & & \uparrow b_{\mu,\lambda} \\ i(\text{Sd}(D_\mu, \mathcal{C}'_\mu)) \times [0, 1] & \xrightarrow{H_\mu} & i(\text{Sd}(D_\mu, \mathcal{C}_\mu)). \end{array}$$

(Note that the first factor of the map on the left hand side of (29) is induced by (26) under the barycentric subdivision functor. Such an abuse of notation also appears in (31) below.)

Indeed, once the commutativity of (29) is established, we will obtain a well-defined deformation retraction

$$H : i(\text{Sd}(X, \mathcal{C}')) \times [0, 1] \longrightarrow i(\text{Sd}(X, \mathcal{C}'))$$

of $i(\text{Sd}(X, \mathcal{C}'))$ onto $i(\text{Sd}(X, \mathcal{C}))$, proving the theorem in view of (27).

The construction of the homotopies (28) is done by induction on $k = \dim e_\lambda$. In the grounding case $k = \min\{\dim e_\mu\}$, each H_λ is defined to be a contraction of D_λ onto its origin $\{0\}$. Suppose we have constructed a homotopy H_μ for each i -cell e_μ with $i \leq k-1$ having the above property. Let $\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda}$ by a k -cell. For a cell $e_\mu \subset \partial e_\lambda$, consider the strong deformation retraction $H_{\mu,\lambda}$ of $b_{\mu,\lambda}(i(\text{Sd}(D_\mu, \mathcal{C}'_\mu)))$ onto $b_{\mu,\lambda}(i(\text{Sd}(D_\mu, \mathcal{C}_\mu)))$ given by the composition

$$\begin{aligned} b_{\mu,\lambda}(i(\text{Sd}(D_\mu, \mathcal{C}'_\mu))) \times [0, 1] &\xrightarrow{b_{\mu,\lambda}^{-1} \times 1} i(\text{Sd}(D_\mu, \mathcal{C}'_\mu)) \times [0, 1] \\ &\xrightarrow{H_\mu} i(\text{Sd}(D_\mu, \mathcal{C}'_\mu)) \xrightarrow{b_{\mu,\lambda}} b_{\mu,\lambda}(i(\text{Sd}(D_\mu, \mathcal{C}'_\mu))). \end{aligned}$$

Notice from Lemma 6.1.13 that we have a cellular stratification

$$\partial D_\lambda = \bigcup_{e_\mu \subset \partial e_\lambda} b_{\mu,\lambda}(\text{Int}(D_\mu))$$

which coincides with the cellular stratification $\mathcal{C}_\lambda|_{\partial D_\lambda}$. Let us show that

$$(30) \quad i(\text{Sd}(\partial D_\lambda, \mathcal{C}'_\lambda|_{\partial D_\lambda})) = \bigcup_{e_\mu \subset \partial e_\lambda} b_{\mu,\lambda}(i(\text{Sd}(D_\mu, \mathcal{C}'_\mu))).$$

Since $b_{\mu,\lambda} : (D_\mu, \mathcal{C}'_\mu) \rightarrow (\partial D_\lambda, \mathcal{C}'_\lambda|_{\partial D_\lambda})$ is an embedding of cellular stratified spaces, there is an induced map

$$(31) \quad b_{\mu,\lambda} : i(\text{Sd}(D_\mu, \mathcal{C}'_\mu)) \longrightarrow i(\text{Sd}(\partial D_\lambda, \mathcal{C}'_\lambda|_{\partial D_\lambda})).$$

between the embedded complexes. Note that (31) is an embedding since so is $b_{\mu,\lambda} : D_\mu \rightarrow \partial D_\lambda$. Thus

$$(32) \quad \bigcup_{e_\mu \subset \partial e_\lambda} b_{\mu,\lambda}(i(\text{Sd}(D_\mu, \mathcal{C}'_\mu))) \subset i(\text{Sd}(\partial D_\lambda, \mathcal{C}'_\lambda|_{\partial D_\lambda})).$$

Conversely, simplices in $\text{Sd}(\partial D_\lambda, \mathcal{C}'_\lambda|_{\partial D_\lambda})$ are in one-to-one correspondence with nondegenerate chains in the face poset $F(\partial D_\lambda, \mathcal{C}'_\lambda|_{\partial D_\lambda})$. For an m -chain $e_0 < \dots < e_m$ in this face poset, there exists a cell e_μ in (X, \mathcal{C}') with $e_\mu \subset \partial e_\lambda$ and $b_{\mu,\lambda}(\text{Int}(D_\mu)) = e_m$. Then this m -chain can be regarded as an m -chain in $(D_\mu, \mathcal{C}'_\mu)$, and we have equality in (32).

Now, the commutativity of (29) for cells $e_{\mu'} \subset \overline{e_\mu} \subset \overline{e_\lambda}$ guarantees that the maps $H_{\mu,\lambda}$ can be glued together to yield a homotopy

$$(33) \quad \bigcup_{e_\mu \subset \partial e_\lambda} H_{\mu,\lambda} : i(\text{Sd}(\partial D_\lambda, \mathcal{C}'_\lambda|_{\partial D_\lambda})) \times [0, 1] \longrightarrow i(\text{Sd}(\partial D_\lambda, \mathcal{C}'_\lambda|_{\partial D_\lambda}))$$

deforming $i(\text{Sd}(\partial D_\lambda, \mathcal{C}'_\lambda|_{\partial D_\lambda}))$ onto $\bigcup_{e_\mu \subset \partial e_\lambda} b_{\mu,\lambda}(i(\text{Sd}(D_\mu, \mathcal{C}'_\mu)))$. But the argument above giving (30) can also be used to show that the latter space agrees with $i(\text{Sd}(\partial D_\lambda, \mathcal{C}_\lambda|_{\partial D_\lambda}))$. Thus we obtain a deformation retraction of $i(\text{Sd}(\partial D_\lambda, \mathcal{C}'_\lambda|_{\partial D_\lambda}))$ onto $i(\text{Sd}(\partial D_\lambda, \mathcal{C}_\lambda|_{\partial D_\lambda}))$.

By construction, we have an identification

$$i(\text{Sd}(D_\lambda, \mathcal{C}'_\lambda)) = \{t\mathbf{x} \mid t \in [0, 1], \mathbf{x} \in i(\text{Sd}(\partial D_\lambda, \mathcal{C}'_\lambda|_{\partial D_\lambda}))\}$$

as a subspace of D_λ , which allows us to define the required homotopy (28) by linearly extending the homotopy in (33).

Naturality of the above construction fails due to the choice of common subdivisions made so that (26) is an embedding of cellular stratified spaces. But the required subdivisions can be made fine enough so to be compatible with respect to a given *finite* set of embeddings and/or isomorphisms (and their compositions). This yields the G -equivariance of the constructions when G is finite. \square

7. THE Σ_n -EQUIVARIANT HOMOTOPY MODEL OF $C_n(S^k)$

This final section is devoted to the proof of Theorem 5.2: by using the method in [BZ92, DS00] (stratifications of euclidean spaces induced by braid arrangements, concepts recalled in the first half of the section), we construct (in the second half of the section) a Σ_n -equivariant cellular homotopy model of $C_n(S^k)$.

7.1. Braid stratifications of euclidean spaces. The following is a direct consequence of Theorems 6.2.4 and 6.2.11.

Corollary 7.1.1. *Let (X, A) be a relatively regular pair of cellular stratified spaces. If $X - A$ is either totally normal or, else, finite and polyhedrally normal, then $B(F(X) - F(A))$ can be embedded in $X - A$ as a strong deformation retract. If in addition a (finite, in the polyhedral normality case) group G acts cellularly on the stratified pair (X, A) , then the strong deformation retraction can be taken to be G -equivariant.*

Our main interest lies in configuration spaces, for which the following special case of Corollary 7.1.1 is fundamental.

Corollary 7.1.2. *Let X be a topological space and let \mathcal{C} be a cellular stratification on X^n under which the fat diagonal*

$$\Delta_n(X) = \{(x_1, \dots, x_n) \in X^n \mid x_i = x_j \text{ for some pair } i, j \text{ with } i \neq j\}$$

is a stratified subspace. Assume further that the pair $(X^n, \Delta_n(X))$ is relatively regular and either totally normal or, else, finite and polyhedrally normal. Let $C_\Delta(\mathcal{C})$ be the induced cellular stratification on the ordered configuration space $C_n(X) = X^n - \Delta_n(X)$. Then $\text{Sd}(C_\Delta(\mathcal{C}))$ is contained in $C_n(X)$ as a strong deformation retract. Furthermore, if $C_\Delta(\mathcal{C})$ is compatible with the Σ_n -action, then $\text{Sd}(C_\Delta(\mathcal{C}))$ is a strong Σ_n -equivariant deformation retract of $C_n(X)$.

The cellular structure \mathcal{C} we use when X is a sphere is motivated by the braid arrangement. Recall that a *hyperplane arrangement* is a finite set of hyperplanes in a finite dimensional real affine space. An especially important arrangement is the *rank $n - 1$ braid arrangement* \mathcal{A}_{n-1} formed by the set of all hyperplanes $x_i - x_j = 0$, $1 \leq i < j \leq n$, in \mathbb{R}^n .

We start by defining the stratification of $\mathbb{R}^n \otimes \mathbb{R}^k$ determined by a general hyperplane arrangement \mathcal{A} in \mathbb{R}^n . The construction, introduced by Björner and Ziegler in [BZ92], requires the concept of higher dimensional sign vectors.

Definition 7.1.3. Consider the set $S_k = \{0, \pm e_1, \dots, \pm e_k\}$. The k -dimensional *sign vector* is the function $\text{sign}_k : \mathbb{R}^k \rightarrow S_k$ given on a tuple $\mathbf{x} = (x_1, \dots, x_k)$ by

$$\text{sign}_k(\mathbf{x}) = \begin{cases} \text{sign}(x_k)e_k & \text{if } \text{sign}(x_k) \neq 0, \\ \text{sign}(x_{k-1})e_{k-1} & \text{if } \text{sign}(x_{k-1}) \neq 0 = \text{sign}(x_k), \\ \dots & \\ \text{sign}(x_1)e_1 & \text{if } \text{sign}(x_1) \neq 0 = \text{sign}(x_i), 2 \leq i \leq k, \\ 0 & \text{if } \mathbf{x} = 0. \end{cases}$$

Definition 7.1.4. Let $\mathcal{A} = \{H_1, \dots, H_q\}$ be a hyperplane arrangement in a real vector space V , where each H_i is defined as the zero set of an affine 1-form $\ell_i : V \rightarrow \mathbb{R}$. These 1-forms yield an affine map $L : V \rightarrow \mathbb{R}^q$ with coordinates ℓ_i , $i = 1, \dots, q$. We tensor L by \mathbb{R}^k and get the map

$$L \otimes \mathbb{R}^k : V \otimes \mathbb{R}^k \rightarrow \mathbb{R}^q \otimes \mathbb{R}^k = (\mathbb{R}^k)^q.$$

Consider the composite

$$\text{sign}_{\mathcal{A} \otimes \mathbb{R}^k} := (\text{sign}_k)^q \circ (L \otimes \mathbb{R}^k) : V \otimes \mathbb{R}^k \longrightarrow S_k^q.$$

The stratification of $V \otimes \mathbb{R}^k$ given by this map,

$$V \otimes \mathbb{R}^k = \coprod_{\mathbf{s} \in \text{Im}(\text{sign}_{\mathcal{A} \otimes \mathbb{R}^k})} (\text{sign}_{\mathcal{A} \otimes \mathbb{R}^k})^{-1}(\mathbf{s}),$$

is called the k^{th} *Björner-Ziegler stratification* associated with \mathcal{A} and is denoted by $\mathcal{C}_{\mathcal{A}}^k$.

Lemma 7.1.5. *Let \mathcal{A} be a real central essential⁴ hyperplane arrangement in a real vector space V . Then it is possible to choose characteristic maps so that $\mathcal{C}_{\mathcal{A}}^k$ is a finite, regular, and totally normal cellular stratification on $V \otimes \mathbb{R}^k$.*

Proof. By definition, cells in this stratification are defined by linear equations and strict linear inequalities, which means that each cell in this stratification is an open convex polyhedron bounded by lower dimensional open convex polyhedra (see Lecture 2 in Ziegler's book [Zi95] for more details). Since \mathcal{A} is central and essential, as is stated in the case $k = 2$ in [BZ92], all cells are open convex cones with vertex at the origin. Furthermore it induces a CW decomposition of the unit disk D of $V \otimes \mathbb{R}^k$ centered at the origin. Hence we obtain a cellular stratification of the interior $\text{Int}(D)$. The standard homeomorphism from $V \otimes \mathbb{R}^k$ to $\text{Int}(D)$ induces an isomorphism of stratified spaces. Cell structures on cells in $\text{Int}(D)$, therefore, induce those on cells in $V \otimes \mathbb{R}^k$. The resulting cellular stratification is obviously regular and totally normal. \square

⁴A hyperplane arrangement is called *central* if the hyperplanes are linear subspaces. A central hyperplane arrangement is called *essential* if the normal vectors to the hyperplanes span the ambient vector space.

Note that $\mathcal{C}_{\mathcal{A}}^k$ is designed to include $\bigcup_i H_i \otimes \mathbb{R}^k$ as a stratified subspace. Thus it also includes the complement

$$M(\mathcal{A} \otimes \mathbb{R}^k) := V \otimes \mathbb{R}^k - \bigcup_i H_i \otimes \mathbb{R}^k$$

as a stratified subspace.

Definition 7.1.6. The induced stratification on $M(\mathcal{A} \otimes \mathbb{R}^k)$ is denoted by $\mathcal{C}_{\mathcal{A}}^{k,\text{comp}}$. The classifying space (order complex) of the face poset of $\mathcal{C}_{\mathcal{A}}^{k,\text{comp}}$, i.e. $\text{Sd}(\mathcal{C}_{\mathcal{A}}^{k,\text{comp}})$, is called the k^{th} order Salvetti complex of \mathcal{A} and is denoted by $\text{Sal}^{(k)}(\mathcal{A})$.

Remark 7.1.7. When $k = 2$, we obtain the classical Salvetti complex constructed by Salvetti in [Sa87]. Note, however, that the term ‘Salvetti complex’ is used for a CW complex $\text{Sal}(\mathcal{A})$ whose face poset $F(\text{Sal}(\mathcal{A}))$ is isomorphic to that of $\mathcal{C}_{\mathcal{A}}^{2,\text{comp}}$. In other words, $\text{Sal}^{(2)}(\mathcal{A})$ is the barycentric subdivision of the standard Salvetti complex.

As a corollary to Theorem 6.2.4, we obtain the following result, which first appeared in the paper [BZ92] by Björner and Ziegler.

Corollary 7.1.8. *Let $\mathcal{A} = \{H_1, \dots, H_q\}$ be a real hyperplane arrangement in a real vector space V . Then $\text{Sal}^{(k)}(\mathcal{A})$ can be embedded into the complement $M(\mathcal{A} \otimes \mathbb{R}^k)$ as a strong deformation retract.*

Remark 7.1.9. The moral is, then, that the general form of Theorem 6.2.4 provides us with a unified framework for working with complements of hyperplane arrangements as well as with configuration spaces (yet, as suggested in the remark at the end of the introduction, the possibilities seem much wider). In fact, our proof of Theorem 5.2 at the end of the section takes advantage, through Corollary 7.1.11 below, of the natural connection (recalled in Example 7.1.12 below) between hyperplane arrangements and configuration spaces of euclidean spaces.

A detailed analysis of the stratification $\mathcal{C}_{\mathcal{A}}^k$, including a proof of Corollary 7.1.8 can be found in the paper [DS00] by De Concini and Salvetti. In particular, their Theorem 1.4.7.(v) determines the dimension of $\text{Sal}^{(k)}(\mathcal{A})$ as follows:

Proposition 7.1.10. *Let \mathcal{A} be a real central essential arrangement in a real vector space of dimension d . Then we have*

$$\dim(\text{Sal}^{(k)}(\mathcal{A})) = d(k - 1).$$

Although \mathcal{A}_{n-1} is not essential, we can apply Proposition 7.1.10 to the essential arrangement \mathcal{A}'_{n-1} given as the restriction of \mathcal{A}_{n-1} to the hyperplane V_n determined by the linear equation $x_1 + \dots + x_n = 0$. Note that the restriction process does not lose any combinatorial information since the linear inclusion $V_n \hookrightarrow \mathbb{R}^n$ induces an inclusion of cellular stratified spaces

$$(V_n \otimes \mathbb{R}^k, \mathcal{C}_{\mathcal{A}'_{n-1}}^k) \hookrightarrow (\mathbb{R}^n \otimes \mathbb{R}^k, \mathcal{C}_{\mathcal{A}_{n-1}}^k)$$

for which the restricted map

$$\left(M(\mathcal{A}'_{n-1} \otimes \mathbb{R}^k), \mathcal{C}_{\mathcal{A}'_{n-1}}^{k, \text{comp}}\right) \hookrightarrow \left(M(\mathcal{A}_{n-1} \otimes \mathbb{R}^k), \mathcal{C}_{\mathcal{A}_{n-1}}^{k, \text{comp}}\right)$$

yields an isomorphism of face posets. Consequently $\text{Sal}^{(k)}(\mathcal{A}_{n-1})$ is simplicially isomorphic to $\text{Sal}^{(k)}(\mathcal{A}'_{n-1})$, and we get:

Corollary 7.1.11. $\dim(\text{Sal}^{(k)}(\mathcal{A}_{n-1})) = (n-1)(k-1)$.

We now have all the ingredients for a proof of Theorem 5.2, but before we start assembling all the pieces, we illustrate the basic building block by recalling, in the following example, the well-known Salvetti complex approach to configuration spaces of points in a euclidean space.

Example 7.1.12. A straightforward check shows that the configuration space $C_n(\mathbb{R}^k)$ agrees with the complement $M(\mathcal{A}_{n-1} \otimes \mathbb{R}^k)$. Corollary 7.1.8 then claims that the k^{th} Salvetti complex $\text{Sal}^{(k)}(\mathcal{A}_{n-1})$ sits inside $C_n(\mathbb{R}^k)$ as a strong deformation retract. Furthermore, the Σ_n -action on $\mathbb{R}^n \otimes \mathbb{R}^k$ is cellular (with respect to $\mathcal{C}_{\mathcal{A}_{n-1}}^k$), and the fat diagonal (which is identified with

$$\bigcup_{1 \leq i < j \leq n} H_{i,j} \otimes \mathbb{R}^k$$

where $H_{i,j}$ stands for the hyperplane $x_i - x_j = 0$ in \mathbb{R}^n) is a Σ_n -invariant subspace. Therefore the final assertion in Corollary 7.1.2 applies, giving a corresponding simplicial complex contained in the unordered configuration space $B_n(\mathbb{R}^k)$ as a strong deformation retract. The important observation here is that Corollary 7.1.11 implies that both models above are dimensionally optimal since, as explained in Remark 5.3 (for spheres rather than euclidean spaces), well-known cohomological calculations of $C_n(\mathbb{R}^k)$ yield in fact

$$(34) \quad \text{hdim}(C_n(\mathbb{R}^k)) = \text{hdim}(B_n(\mathbb{R}^k)) = (n-1)(k-1).$$

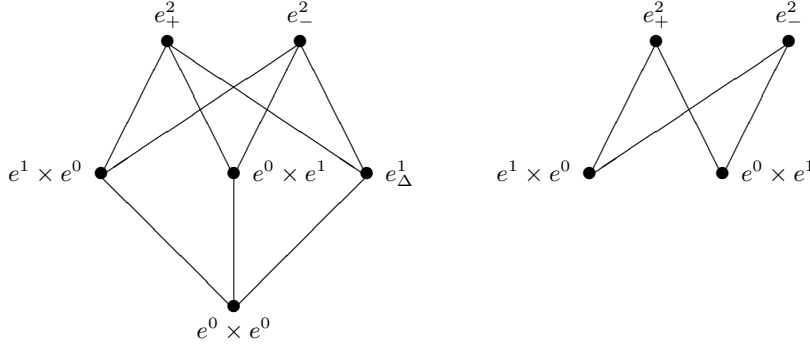
7.2. Braid stratifications of products of a sphere. The remainder of the paper can be thought of as adapting the considerations in Example 7.1.12 to the case of configuration spaces on spheres. For starters, the next two examples work out in full detail the situation for configuration spaces of two distinct points on the circle (Example 7.2.1) and on the 2-sphere (Example 7.2.2).

Example 7.2.1. Let $S^1 = e^0 \cup e^1$ be the minimal CW complex decomposition, and consider the corresponding product decomposition in $S^1 \times S^1$. Since this does not contain the diagonal $\Delta_2(S^1)$ as a stratified subspace, we subdivide $e^1 \times e^1$ along the diagonal. The resulting cellular stratification $\mathcal{B}_{1,2}$ on $S^1 \times S^1$ is

$$S^1 \times S^1 = e^0 \times e^0 \cup e^0 \times e^1 \cup e^1 \times e^0 \cup e_{\Delta}^1 \cup e_+^2 \cup e_-^2,$$

with Hasse diagram⁵ shown on the left hand side of the following figure:

⁵Recall that the *Hasse diagram* of a poset is the graph whose vertices are elements of the poset, where two vertices x, y are connected by an edge if $x < y$ and there is no element z with $x < z < y$. Elements are ordered from bottom to top, starting with minimal ones.



Although $\mathcal{B}_{1,2}$ is not regular (but it is strongly normal), it is easy to see that the pair $(S^1 \times S^1, \Delta_2(S^1))$ is relatively regular and totally normal. Further,

$$F(C_\Delta(\mathcal{B}_{1,2})) = F(S^1 \times S^1) - F(\Delta_2(S^1))$$

is the subposet of $F(S^1 \times S^1)$ obtained by removing the cells $e^0 \times e^0$ and e^1_Δ . The corresponding Hasse diagram, shown on the right hand side of the figure above, is obtained by removing the corresponding vertices together with those edges having these vertices as one of their ends. Note that this is the Hasse diagram of the minimal Σ_2 -equivariant CW complex decomposition of S^1 , i.e. $S^1 = e^0_+ \cup e^0_- \cup e^1_+ \cup e^1_-$, so that

$$\text{Sd}(C_\Delta(\mathcal{B}_{1,2})) = BF(C_\Delta(\mathcal{B}_{1,2})) \cong_{\Sigma_2} S^1.$$

In view of Corollary 7.1.2, this gives a direct combinatorial explanation of the case $n = 2$ in (15).

The situation for the 2-sphere is more involved, and should be thought of as the complexified version of Example 7.2.1:

Example 7.2.2. Consider the product decomposition

$$(35) \quad S^2 \times S^2 = e^0 \times e^0 \cup e^0 \times e^2 \cup e^2 \times e^0 \cup e^2 \times e^2$$

coming from the minimal CW complex decomposition $S^2 = e^0 \cup e^2$. This time the diagonal in $e^2 \times e^2$ does not divide $e^2 \times e^2$ into pieces; instead the required subdivision arises from a direct comparison with $\mathcal{C}_{\mathcal{A}_1}^2$. In terms of the identification

$$e^2 \times e^2 \cong \mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{C} \times \mathbb{C},$$

the diagonal corresponds to the complexification of the braid arrangement \mathcal{A}_1 . The associated Björner-Ziegler stratification on \mathbb{C}^2 is given by

$$\begin{aligned} \mathbb{C}^2 = & \{ (z_1, z_2) \in \mathbb{C}^2 \mid z_1 = z_2 \} \\ & \cup \{ (z_1, z_2) \in \mathbb{C}^2 \mid \text{Im}(z_1 - z_2) = 0, \text{Re}(z_1 - z_2) > 0 \} \\ & \cup \{ (z_1, z_2) \in \mathbb{C}^2 \mid \text{Im}(z_1 - z_2) = 0, \text{Re}(z_1 - z_2) < 0 \} \\ & \cup \{ (z_1, z_2) \in \mathbb{C}^2 \mid \text{Im}(z_1 - z_2) > 0 \} \\ & \cup \{ (z_1, z_2) \in \mathbb{C}^2 \mid \text{Im}(z_1 - z_2) < 0 \} \end{aligned}$$

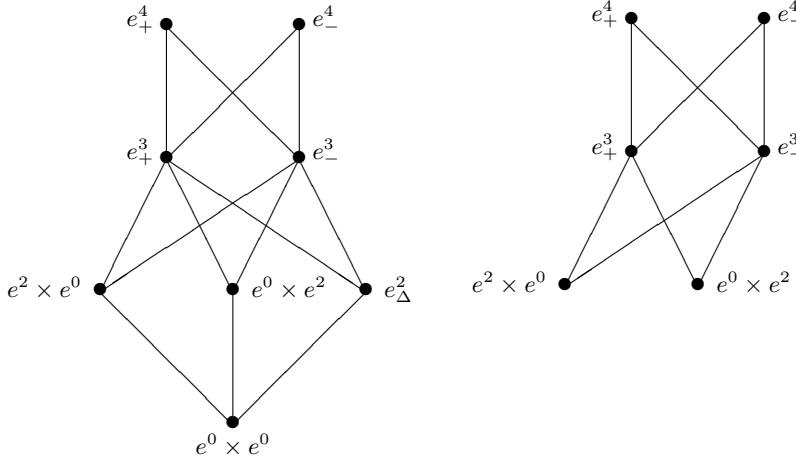
which yields a cellular stratification

$$e^2 \times e^2 = e_{\Delta}^2 \cup e_+^3 \cup e_-^3 \cup e_+^4 \cup e_-^4$$

and, in terms of (35), a CW complex decomposition $\mathcal{B}_{2,2}$ on $S^2 \times S^2$. Since the two cells $e^0 \times e^0$ and e_{Δ}^2 give a stratification of the diagonal $\Delta_2(S^2)$, the cellular stratified space $C_{\Delta}(\mathcal{B}_{2,2})$ takes the form

$$C_2(S^2) = e^0 \times e^2 \cup e^2 \times e^0 \cup e_+^3 \cup e_-^3 \cup e_+^4 \cup e_-^4.$$

Note that the Hasse diagrams of $F(\mathcal{B}_{2,2})$ and $F(C_{\Delta}(\mathcal{B}_{2,2}))$ are respectively given by



Therefore $F(C_{\Delta}(\mathcal{B}_{2,2}))$ is isomorphic to the face poset of the minimal Σ_2 -equivariant regular CW complex decomposition of S^2 , so that

$$\text{Sd}(C_{\Delta}(\mathcal{B}_{2,2})) = BF(C_{\Delta}(\mathcal{B}_{2,2})) \cong_{\Sigma_2} S^2.$$

Thus, just as in Example 7.2.1, Corollary 7.1.2 gives a direct combinatorial explanation for the case $k = 2$ in (14). Note, however, that the verification of the relative regularity and total normality of $C_{\Delta}(\mathcal{B}_{2,2})$ is a more involved matter than in the case of $C_{\Delta}(\mathcal{B}_{1,2})$ in Example 7.2.1. For the general case, such a task is accomplished (in a suitably weaker—but sufficient—form) in Proposition 7.2.4 below by making use of Theorem 6.1.14.

The above examples suggest that a conveniently controlled stratification on $C_n(S^k)$ can be obtained by using the k^{th} Björner-Ziegler cellular stratification associated with the braid arrangement \mathcal{A}_{j-1} . The idea is to subdivide each jk -dimensional cell in the product CW complex decomposition of $(S^k)^n$ coming from the minimal CW structure of S^k . The finer CW decomposition we need is described in full detail next.

Definition 7.2.3. Consider the minimal CW complex decomposition $S^k = e^0 \cup e^k$, and let

$$(36) \quad \varphi_k: I^k \rightarrow S^k$$

be the characteristic map of e^k given by collapsing to a point the boundary of I^k , where $I = [-1, 1]$. Each cell e in the corresponding product decomposition of the n -fold cartesian

product $(S^k)^n$ can be identified with the m -fold cartesian product $(e^k)^m$ for some $m = m(e)$, $0 \leq m \leq n$, and the m -fold product of (36),

$$\varphi_{k,m}: I^{km} = (I^k)^m \rightarrow (S^k)^m,$$

can be regarded as the characteristic map of $e \cong (e^k)^m$. In terms of the identifications $\text{Int}(I^{km}) \cong \mathbb{R}^{km} \cong \mathbb{R}^m \otimes \mathbb{R}^k$ (where the first homeomorphism is given by radial expansion from the origin), $\mathcal{C}_{\mathcal{A}_{m-1}}^k$ induces a cellular stratification on $\text{Int}(I^{km})$ and, hence, on $e \cong (e^k)^m$. Let us denote the resulting stratification by

$$(37) \quad (e^k)^m = \bigcup_{\lambda \in \Lambda_m} e_{m,\lambda}.$$

Note that the stratification on $\text{Int}(I^{km})$ can be alternatively defined by restricting $\mathcal{C}_{\mathcal{A}_{m-1}}^k$ under the canonical inclusion $\text{Int}(I^{km}) \subset \mathbb{R}^{km}$ (indeed, radial expansion from the origin is stable on a given cell). Therefore, the stratification extends to a structure of polyhedral complex $P_{k,m}$ on I^{km} . Now, for each $\lambda \in \Lambda_m$, let $D_{m,\lambda}$ be the closure of $\varphi_{k,m}^{-1}(e_{m,\lambda})$ in I^{km} . Then $D_{m,\lambda}$ is a polyhedral face in $P_{k,m}$, and is homeomorphic to a closed disk (of dimension that of $e_{m,\lambda}$). Thus the restriction

$$(38) \quad \varphi_{m,\lambda} = \varphi_{k,m}|_{D_{m,\lambda}}: D_{m,\lambda} \rightarrow \overline{e_{m,\lambda}}$$

defines a cell structure on $e_{m,\lambda}$. The resulting CW decomposition on $(S^k)^n$, denoted by $\mathcal{B}_{k,n}$, is called the *braid stratification on $(S^k)^n$* .

Proposition 7.2.4. *The CW complex $((S^k)^n, \mathcal{B}_{k,n})$:*

- (i) *is cellularly compatible with the Σ_n -action on $(S^k)^n$;*
- (ii) *has the fat diagonal $\Delta_n(S^k)$ as a stratified subspace;*
- (iii) *is locally polyhedral;*
- (iv) *yields a finite, regular, and polyhedrally normal cellular stratification on $C_n(S^k)$.*

Proof. The first two assertions follow from a standard verification—similar to the one needed at the start of Example 7.1.12. On the other hand, the polyhedral normality clause in the fourth assertion follows from the regularity clause, the third assertion, and Theorem 6.1.14. Therefore, it suffices to check (iii) and the regularity property asserted in (iv).

Building on the notation in (37), the cell decomposition $\mathcal{B}_{k,n}$ is

$$(S^k)^n = \bigcup_{m=0}^n \bigcup_{\sigma \in \Sigma_{n,m}} \bigcup_{\lambda \in \Lambda_m} \sigma(e_{m,\lambda})$$

where we interpret $\Sigma_{n,m} = \{(i_1, \dots, i_m) \mid 1 \leq i_1 < \dots < i_m \leq n\}$ as the set of “shuffle inclusions” from $(S^k)^m$ to $(S^k)^n$, i.e. an element $\sigma = (i_1, \dots, i_m)$ in $\Sigma_{n,m}$ is regarded as the inclusion $\sigma: (S^k)^m \hookrightarrow (S^k)^n$ that maps the ℓ -th coordinate to the i_ℓ -th coordinate, putting the base point e^0 in the remaining coordinates.

In order to verify the normality of $\mathcal{B}_{k,n}$ (required by (iii) in view of Definition 6.1.8), it is enough to check the normality in $\mathcal{B}_{k,m}$ of each cell $e_{m,\lambda}$ —indeed, each inclusion $\sigma: (S^k)^m \hookrightarrow (S^k)^n$ is cellular and (topologically) closed. In fact, since

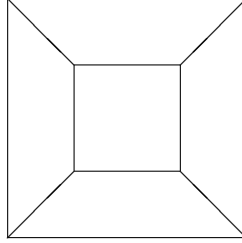
$$(39) \quad \partial D_{m,\lambda} = \bigcup_{\phi \in \Phi} \phi$$

is a union of codimension-1 faces of the polyhedral disk $D_{m,\lambda}$, it suffices to show that the $\varphi_{m,\lambda}$ -image of each $\phi \in \Phi$ is a union of (a) cells in $\mathcal{B}_{k,m}$ and (b) closures of cells in $\mathcal{B}_{k,m-1}$. Now, when a face $\phi \in \Phi$ is included in ∂I^{km} (an “external” face), i.e. if the defining equations of ϕ contain an equality of the form

$$(40) \quad x_{i_\ell} = \pm 1,$$

then the $\varphi_{m,\lambda}$ -image of ϕ is, up to a shuffle, the product of e^0 with a union of closures of cells in $\mathcal{B}_{k,m-1}$. On the other hand, removing ∂I^{km} from a non-external face $\phi \in \Phi$ yields a face created by the (normal, in view of Lemma 7.1.5) stratification $\mathcal{C}_{\mathcal{A}_{m-1}}^k$, so that the difference of sets $\phi - \partial I^{km}$ is a union of cells in $\mathcal{B}_{k,m}$. Normality of $\mathcal{B}_{k,n}$ thus follows.

In order to produce the locally polyhedral structure asserted in (iii), we start by considering the k -dimensional polyhedral complex Q_k obtained by removing the interior of the top face of ∂I^{k+1} , and flattening—in k dimensions—the resulting complex (as suggested in the following picture where $k = 2$).



Under the identification of S^k with the boundary of the cone on I^k , $\partial C(I^k)$, a polyhedral replacement of (36) is given by the obvious homeomorphism $\alpha: Q_k \cong I^k$. Indeed, the composite

$$Q_k \xrightarrow{\alpha} I^k \xrightarrow{\varphi_k} S^k \cong \partial C(I^k),$$

given by collapsing to a point the outer square in the above picture, is a PL map. More generally, for an element $\sigma = (i_1, i_2, \dots, i_m) \in \Sigma_{n,m}$, the composite

$$(41) \quad I^{km} \xrightarrow{\varphi_{k,m}} (\partial C(I^k))^m \xrightarrow{\sigma} (\partial C(I^k))^n$$

is the characteristic map of an km -cell in $(\partial C(I^k))^n$ with a polyhedral replacement given by $\alpha^m: Q_k^m \rightarrow I^{km}$. Now, as we observed in Definition 7.2.3, the braid arrangement \mathcal{A}_{m-1} induces a polyhedral decomposition $P_{k,m}$ on $I^{km} = (I^k)^m$. By taking a common subdivision of $P_{k,m}$ and Q_k^m , we obtain a polyhedral decomposition of each $D_{m,\lambda}$ under which the restriction to $D_{m,\lambda}$ of (41) is a PL map. Thus we obtain the required locally polyhedral structure.

It remains to check the regularity property asserted in (iv). A cell e in $(S^k)^n$ is a product of e^0 's and e^k 's. If e^0 occurs as a factor more than once, then e belongs to $\Delta_n(S^k)$ and does not contribute anything to $C_\Delta(\mathcal{B}_{k,n})$. Thus, instead of $(S^k)^n$, we start with the stratified subspace of $(S^k)^n$ given by

$$X_{k,n} = (e^k)^n \cup \bigcup_{\ell} ((e^k)^{\ell-1} \times e^0 \times (e^k)^{n-\ell}).$$

Our stratified space $(C_n(S^k), C_\Delta(\mathcal{B}_{k,n}))$ is obtained from $X_{k,n}$ by subdividing each cell via \mathcal{A}_{n-1} (in the case of the top dimensional cell of $X_{k,n}$) or \mathcal{A}_{n-2} (in the case of the remaining cells), and then removing cells in the fat diagonal. As in (37), we denote the Björner-Ziegler cellular stratification on the top cell in $X_{k,n}$ by

$$(42) \quad (e^k)^n = \bigcup_{\lambda \in \Lambda_n} e_{n,\lambda}$$

and, for the lower layer of cells, we adapt the notation to

$$(43) \quad (e^k)^{\ell-1} \times e^0 \times (e^k)^{n-\ell} = \bigcup_{\mu \in \Lambda_{n-1}} e_{n-1,\ell,\mu}.$$

The regularity of cells of the form $e_{n-1,\ell,\mu}$ is a direct consequence of Lemma 7.1.5. On the other hand, the characteristic map (in $C_\Delta(\mathcal{B}_{k,n})$) of a cell of the form $e_{n,\lambda}$ is a restriction of (38)—with $m = n$. Namely, the domain $D'_{n,\lambda}$ of $e_{n,\lambda}$ as a cell of $C_\Delta(\mathcal{B}_{k,n})$ is obtained from $D_{n,\lambda}$ by removing all codimension-2 faces ψ in ∂I^{km} . Thus, the analysis in the paragraph containing (40) shows that the corresponding characteristic map $\varphi'_{n,\lambda}$ becomes a homeomorphism. Indeed, for $\phi \in \Phi$ as in (39), the $\varphi'_{m,\lambda}$ -image of each $\phi - \psi$ is a union of (a) cells in $\mathcal{B}_{k,n}$ and (b) cells in $\mathcal{B}_{k,n-1}$ (no closures needed now—compare with the considerations following (39)), and in either case $\varphi'_{n,\lambda}$ is a homeomorphism. \square

Corollary 7.1.2 and Proposition 7.2.4 imply that $\text{Sd}(C_\Delta(\mathcal{B}_{k,n}))$ can be embedded in $C_n(S^k)$ as a strong Σ_n -equivariant deformation retract. The dimension of $\text{Sd}(C_\Delta(\mathcal{B}_{k,n}))$ can be computed by comparing with the face poset of the Salvetti complex for the braid arrangement. The following explicit description implies Theorem 5.2.

Theorem 7.2.5. *$\text{Sd}(C_\Delta(\mathcal{B}_{k,n}))$ is a simplicial complex of dimension $d(k,n)$ embedded in $C_n(S^k)$ as a strong Σ_n -equivariant deformation retract.*

Proof. We complete the only remaining task (i.e. counting the dimension of $\text{Sd}(C_\Delta(\mathcal{B}_{k,n}))$) by using the strategy at the end of the proof of Proposition 7.2.4: we focus on the two kinds of cells in (42) and (43).

The braid arrangement \mathcal{A}_{n-2} gives rise to a cellular stratification of the cells in (43). The top cells in the resulting stratification (which are also cells of $C_\Delta(\mathcal{B}_{k,n})$) are evidently in dimension $k(n-1)$. On the other hand, Corollary 7.1.11 implies that the minimal dimension of cells in $C_\Delta(\mathcal{B}_{k,n})$ coming from the various \mathcal{A}_{n-2} stratifications is

$$k(n-1) - \dim(\text{Sal}^{(k)}(\mathcal{A}_{n-2})) = k(n-1) - (k-1)(n-2) = k+n-2.$$

Likewise, the cells of $C_\Delta(\mathcal{B}_{k,n})$ coming from the \mathcal{A}_{n-1} stratification on the cell in (42) are in dimensions in between kn and

$$kn - \dim(\text{Sal}^{(k)}(\mathcal{A}_{n-1})) = kn - (k-1)(n-1) = k + n - 1.$$

Thus the rank of $F(C_\Delta(\mathcal{B}_{k,n}))$, i.e. the dimension of $\text{Sd}(C_\Delta(\mathcal{B}_{k,n}))$, is given by $kn - k - n + 2 = (k-1)(n-1) + 1 = d(k, n)$. \square

REFERENCES

- [BG61] Bernstein, I.; Ganea, T.: The category of a map and of a cohomology class. *Fund. Math.* **50** (1961/1962) 265–279.
- [BLSWZ99] Björner, A.; Las Vergnas, M.; Sturmfels, B.; White, N.; Ziegler, G. M.: *Oriented matroids*, second edition. Encyclopedia of Mathematics and its Applications **46**, Cambridge University Press, Cambridge, 1999.
- [BZ92] Björner, A.; Ziegler, G. M.: Combinatorial stratification of complex arrangements. *J. Amer. Math. Soc.* **5** (1992) 105–149.
- [Br72] Bredon, G. E.: *Introduction to compact transformation groups*. Pure and Applied Mathematics **46**, Academic Press, New York-London, 1972.
- [CLOT03] Cornea, O.; Lupton, G.; Oprea, J.; Tanré, D.: *Lusternik-Schnirelmann category*. Mathematical Surveys and Monographs **103**, American Mathematical Society, Providence, RI, 2003.
- [DS00] De Concini, C.; Salvetti, M.: Cohomology of Coxeter groups and Artin groups. *Math. Res. Lett.* **7** (2000) 213–232.
- [Do63] Dold, A.: Partitions of unity in the theory of fibrations. *Ann. of Math.* **78** (1963) 223–255.
- [Do95] Dold, A.: *Lectures on algebraic topology*, reprint of the 1972 edition. Classics in Mathematics, Springer-Verlag, Berlin, 1995.
- [ES52] Eilenberg, S.; Steenrod, N.: *Foundations of algebraic topology*. Princeton University Press, Princeton, New Jersey, 1952.
- [Fa03] Farber, M.: Topological complexity of motion planning. *Discrete Comput. Geom.* **29** (2003) 211–221.
- [Fa04] Farber, M.: Instabilities of robot motion. *Topology Appl.* **140** (2004) 245–266.
- [Fa06] Farber, M.: *Topology of robot motion planning*. In: Morse theoretic methods in nonlinear analysis and in symplectic topology, NATO Science Series II: Mathematics, Physics and Chemistry **217**, 185–230, Springer, Dordrecht, 2006.
- [Fa08] Farber, M.: *Invitation to topological robotics*. Zurich Lectures in Advanced Mathematics, EMS, Zürich, 2008.
- [FG07] Farber, M.; Grant, M.: Symmetric motion planning. In: Topology and Robotics, *Contemp. Math.* **438**, 85–104, Amer. Math. Soc., Providence, RI, 2007.
- [FG08] Farber, M.; Grant, M.: Robot motion planning, weights of cohomology classes, and cohomology operations. *Proc. Amer. Math. Soc.* **136** (2008) 3339–3349.
- [FTY03] Farber, M.; Tabachnikov, S.; Yuzvinsky, S.: Topological robotics: motion planning in projective spaces. *Int. Math. Res. Not.* **34** (2003) 1853–1870.
- [FY04] Farber, M.; Yuzvinsky, S.: Topological robotics: subspace arrangements and collision free motion planning. In: Geometry, Topology, and Mathematical Physics, *Amer. Math. Soc. Transl.* (2) **212**, 145–156, Amer. Math. Soc., Providence, RI, 2004.
- [FZ00] Feichtner, E. M.; Ziegler, G. M.: The integral cohomology algebras of ordered configuration spaces of spheres. *Doc. Math.* **5** (2000) 115–139.
- [Fu70] Fuks (Fuchs), D. B.: Cohomology of the braid group mod 2. *Funktsional. Anal. i Prilozhen.* **4** (1970) 62–73.

- [FR84] Fuks (Fuchs), D. B; Rokhlin, V. A.: *Beginner's course in topology. Geometric chapters*. Universitext, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1984.
- [Gl50] Gleason, A.: Spaces with a compact Lie group of transformations. *Proc. Amer. Math. Soc.* **1** (1950) 35–43.
- [GL09] González, J.; Landweber, P.: Symmetric topological complexity of projective and lens spaces. *Algebr. Geom. Topol.* **9** (2009) 473–494.
- [IS10] Iwase, N.; Sakai, M.: Topological complexity is a fibrewise L-S category. *Topology Appl.* **157** (2010) 10–21.
- [J76] Jaworowski, J.: Extensions of G -maps and Euclidean G -retracts. *Math. Z.* **146** (1976) 143–148.
- [Ka08] Kallel, S.: Symmetric products, duality and homological dimension of configuration spaces. *Geom. Topol. Monogr.* **13** (2008) 499–527.
- [KL12] Karasev, R.; Landweber, P.: Estimating the higher symmetric topological complexity of spheres. *Algebraic & Geometric Topology* **12** (2012) 75–94.
- [KV10] Karasev, R. N.; Volovikov, A. Yu.: Configuration-like spaces and coincidences of maps on orbits. arXiv:0911.4338v3 [math.AT].
- [Ki11] Kirillov Jr., A.: On piecewise linear cell decompositions. arXiv: 1009.4227v2 [math.GT].
- [Ko08] Kozlov, D.: *Combinatorial algebraic topology*. Algorithms and Computation in Mathematics **21**, Springer, Berlin, 2008.
- [LV06] LaValle, S.: *Planning algorithms*. Cambridge University Press, Cambridge, 2006.
- [La91] Latombe, J.-C.: *Robot motion planning*. The Kluwer international series in engineering and computer science **124**, Kluwer Academic Publishers, Boston, 1991.
- [LW69] Lundell, A. T.; Weingram, S.: *The topology of CW complexes*. The University Series in Higher Mathematics, Van Nostrand Reinhold Co., New York, 1969.
- [Ro08] Roth, F.: On the category of Euclidean configuration spaces and associated fibrations. *Geom. Topol. Monogr.* **13** (2008) 447–461.
- [Ru10] Rudyak, Yu. B.: On higher analogs of topological complexity. *Topology and its Applications* **157** (2010) 916–920 (erratum in *Topology and its Applications* **157** (2010) 1118).
- [RS72] Rourke, C. P.; Sanderson, B. J.: *Introduction to piecewise-linear topology*. Springer-Verlag, New York, 1972.
- [Sa04] Salvatore, P.: Configuration spaces on the sphere and higher loop spaces. *Math. Z.* **248** (2004) 527–540.
- [Sa87] Salvetti, M.: Topology of the complement of real hyperplanes in \mathbb{C}^N . *Invent. Math.* **88** (1987) 603–618.
- [Sc03] Schürmann, J.: *Topology of singular spaces and constructible sheaves*. Instytut Matematyczny Polskiej Akademii Nauk, Monografie Matematyczne (New Series) **63**, Birkhäuser Verlag, Basel, 2003.
- [Se51] Serre, J-P.: Homologie singulière des espaces fibrés. Applications. *Ann. of Math.* (2) **54** (1951) 425–505.
- [Sm87] Smale, S.: On the topology of algorithms, I. *Journal of Complexity* **3** (1987) 81–89.
- [Sv66] Švarc (Schwarz), A. S.: The genus of a fiber space. *Amer. Math. Soc. Transl. Series 2* **55** (1966) 49–140.
- [Tn11] Tanaka, K.: The cylindrical structure on manifolds via Morse theory arXiv:1106.3374v1 [math.GT].
- [Va88] Vasil'ev (Vassiliev), V. A.: Cohomology of braid groups and the complexity of algorithms. *Funktsional. Anal. i Prilozhen.* **22** (1988) 15–24 (English translation in *Funct. Anal. Appl.* **22** (1989) 182–190).
- [Zi95] Ziegler, G. M.: *Lectures on polytopes*. Graduate Texts in Mathematics **152**, Springer-Verlag, New York, 1995.

IBAI BASABE

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA
358 LITTLE HALL, GAINESVILLE, FL 32611-8105, USA

E-mail address: `iebasabe1@ufl.edu`

JESÚS GONZÁLEZ

DEPARTAMENTO DE MATEMÁTICAS, CINVESTAV-IPN
A.P. 14-740, MÉXICO CITY 07000, MÉXICO

E-mail address: `jesus@math.cinvestav.mx`

YULI B. RUDYAK

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA
358 LITTLE HALL, GAINESVILLE, FL 32611-8105, USA

E-mail address: `rudyak@ufl.edu`

DAI TAMAKI

DEPARTMENT OF MATHEMATICAL SCIENCES, SHINSHU UNIVERSITY
MATSUMOTO, 390-8621, JAPAN

E-mail address: `rivulus@math.shinshu-u.ac.jp`